

The Exact Solution of the Trust Region Quadratic Problem

H.E.K.

The necessary and sufficient conditions for the solution of the spherical trust region quadratic problem are stated in N&W Theorem 4.1, but its proof is not so simple. Actually, the problem is a *constrained optimization problem*, contrary to the unconstrained problems we have looked at so far.

Below is an attempt to summarize the arguments. Recall that we consider

$$\min_{x \in \mathbb{R}^n} f(p).$$

Standing at a point x_k we are in the center of a *trust region*, D , typically a ball with radius Δ , where we "have trust in" an approximation m to f ,

$$f(x_k + p) \approx m(p) = f(x_k) + b'p + \frac{1}{2}p'Bp.$$

It is reasonable to let $b = \nabla f(x_k)$, and B be somewhat similar to $\nabla^2 f(x_k)$. The simplified problem, which is the core of the Trust Region algorithm, is now

$$\min_{p \in D} m(p), \tag{1}$$

where

$$m(p) = b'p + \frac{1}{2}p'Bp, \tag{2}$$

$$D = \{p; \|p\| \leq \Delta\}. \tag{3}$$

The object function is *quadratic* and the domain is a *ball*. The matrix B is assumed to be symmetric (since skew symmetric parts will not contribute to $p'Bp$ in any case!).

Since m is continuous and D is closed and bounded, we know that there *always* exist minima. We also know that

$$\nabla m(p)' = b + Bp, \tag{4}$$

$$\nabla^2 m(p) = B, \tag{5}$$

and the Taylor expansion around a minimum p^* (or any fixed point) has the form

$$m(p^* + \delta) = m(p^*) + \nabla m(p^*)\delta + \frac{1}{2}\delta'B\delta. \tag{6}$$

Note that we do not require, or need, that B is positive semi-definite. This, and the constraint in Eqn. 3 complicates the solution considerably compared to the standard unconstrained quadratic model problem.

1 Necessary and sufficient conditions for minima

We first observe that there are two possibilities:

1. either $\nabla m(p) = 0$ for *some* $p \in D$
2. or $\nabla m(p) \neq 0$ for *all* $p \in D$ (including the boundary of D , ∂D).

Consider the first case, and assume that

$$\nabla m(p^*) = b + Bp^* = 0 \quad (7)$$

for some p^* in the *interior* of D . The second order necessary condition for minima then requires that $B \geq 0$, and by inspecting Eqn. 6, this is actually also *sufficient* for p^* to be a minimum ($B \geq 0$ implies that $m(p)$ is convex!).

We are not quite finished with the first case yet: What if p^* happens to be on ∂D ? Equation 6 still holds, and we must have that $\delta' B \delta \geq 0$, at least for all vectors δ pointing into D . However, $\delta' B \delta = (-\delta)' B (-\delta)$, so that it must actually hold for all δ -s, and hence B *must* be positive semi-definite (*No*, it can not be strictly negative even for vectors that are tangents to the boundary! Prove it, or see the argument in the appendix). On the other hand, $\nabla m(p^*) = 0$ and $B \geq 0$ will also be sufficient conditions for a minimum.

We then look at the second and more difficult case where $\nabla m(p)' = b + Bp$ is non-zero for all $p \in D$. The solution will for sure be somewhere on ∂D . In general, all directions within $\pi/2$ of the negative gradient are descent directions, and since the vector $p^* \in \partial D$ at the same time is an outward normal vector to the boundary, $\nabla m(p^*)$ has to point *exactly* opposite to p^* at a minimum (Make a simple sketch for $\mathbb{R}^2$ if you do not see this!). Thus, p^* is a minimum for the second case *only* if $\nabla m(p^*)$ is proportional to $-p^*$, or, equivalently, there is a $\lambda > 0$ so that

$$b + Bp^* = -\lambda p^*. \quad (8)$$

This may also be written

$$(B + \lambda I)p^* = -b. \quad (9)$$

It is also possible to show that that $B + \lambda I$ will be positive semi-definite, but the argument in N&W is tricky and reproduced in the Appendix at the end.

For the converse, let us assume that we have found a $\lambda > 0$ such that Eqn. 9 holds, $B + \lambda I \geq 0$, and $\|p^*\| = \Delta$. Then the function

$$m_\lambda(p) = m(p) + \frac{\lambda}{2} (p'p - p'^*p^*) \quad (10)$$

is a quadratic, and

$$\nabla m_\lambda(p^*) = b + (B + \lambda I)p^* = 0, \quad (11)$$

$$\nabla^2 m_\lambda(p^*) = B + \lambda I \geq 0. \quad (12)$$

Thus, according to the first case, p^* is a minimum for $m_\lambda(p)$ in D . But

$$m(p^*) = m_\lambda(p^*) \leq m_\lambda(p) = m(p) + \frac{\lambda}{2} (p'p - p'^*p^*) \leq m(p), \quad (13)$$

since $\|p\| \leq \|p^*\|$ for all $p \in D$. Thus, p^* is a minimum for $m(p)$ as well!

In summary:

The vector p^ is an exact solution to the Trust Region quadratic problem if and only if one of the following conditions hold:*

$$\begin{aligned} (i) & \quad b + Bp^* = 0, \quad B \geq 0, \|p^*\| \leq \Delta, \\ (ii) & \quad \text{There exists a } \lambda > 0 \text{ such that } (B + \lambda I)p^* = -b, \quad B + \lambda I \geq 0, \quad \|p^*\| = \Delta. \end{aligned} \quad (14)$$

2 How to find a solution

First of all, we may be lucky and find that (i) holds, e.g. $B > 0$ and $\|B^{-1}(-b)\| \leq \Delta$. If we are not so lucky, perhaps the equation for p^* in (i) cannot be solved, we need to consider (ii). Then there is an extra parameter in our problem, namely λ , which we later will identify as a *Lagrangian* parameter coming from the constraint (Eqn. 3).

In order to analyze these cases, we recall that B was supposed to be symmetric, so it has n real eigenvalues, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, and a corresponding set of normalized, orthogonal eigenvectors $\{e_j\}_{j=1}^n$.

When $\lambda_j + \lambda \neq 0$ for all j , $B + \lambda I$ will be non-singular and we can solve Eqn.9,

$$p_\lambda = (B + \lambda I)^{-1}(-b). \quad (15)$$

Show that the solution in Eqn. 15 may be expressed as

$$p_\lambda = \sum_{j=1}^n \left(\frac{e'_j(-b)}{\lambda_j + \lambda} \right) e_j \quad (16)$$

and hence

$$\|p_\lambda\|^2 = \sum_{j=1}^n \left(\frac{e'_j(-b)}{\lambda_j + \lambda} \right)^2. \quad (17)$$

We have to look for a solution λ^* in the interval $[-\lambda_1, \infty)$, since this will ensure that $B + \lambda I \geq 0$. Moreover, $\|p_{\lambda^*}\| = \Delta$. What happens with $\|p_\lambda\|^2$ when $\lambda \rightarrow \infty$? And, assuming that $e'_1(-b) \neq 0$, when λ approaches $-\lambda_1$ from above? Draw a sketch of the RHS of Eqn.17, or look at Fig.4.5 in N&W. Since the right hand side of Eqn.17 is a continuous function in the interval $(-\lambda_1, \infty)$, there has to be a λ^* so that $\|p_\lambda\| = \Delta$. Finding λ^* thus amounts to find a solution of a non-linear equation. There is one little snag left in this argument: What happens if $B > 0$, so that $-\lambda_1 < 0$? Our solution requires $\lambda^* > 0$. Try to fill in the details by considering p_λ for $\lambda = 0$.

In exceptional cases, $e'_1(-b) = 0$. We then have to solve Eqn.9 on the subspace spanned by $\{e_2, \dots, e_n\}$, or, if necessary, choose $\lambda^* = -\lambda_1$ and add some contribution from e_1 so that $\|p_\lambda\| = \Delta$ (This is *The Hard Case*, p. 87 in N&W).

The above covers much of the discussion in Chapter 4, and we shall leave the general problem here. In practice, it is more reasonable to apply some of the simplified, but approximate solutions to the quadratic problem, as described in N&W §4.1.

Note: Trond Steihaug, referred to on p. 75 is Norwegian, and professor in Informatics/Numerical Analysis at the University of Bergen.

3 Appendix: $B + \lambda I \geq 0$.

We assume that p^* is a minimum for $m(p)$, $\|p^*\| = \Delta$, $\lambda > 0$, and $(B + \lambda I)p^* = -b$. Following N&W, we consider the following expression for $p \in \partial D$ (and note that $p^{*'}p^* - p'p = 0$):

$$\begin{aligned}
0 &\leq m(p) - m(p^*) - \frac{\lambda}{2}(p^{*'}p^* - p'p) \\
&= b'p + \frac{1}{2}p'Bp - b'p^* - \frac{1}{2}p^{*'}Bp^* - \frac{\lambda}{2}(p^{*'}p^* - p'p) \\
&= -p'(B + \lambda I)p^* + \frac{1}{2}p'Bp + p^{*'}(B + \lambda I)p^* - \frac{1}{2}p^{*'}Bp^* - \frac{\lambda}{2}(p^{*'}p^* - p'p) \\
&= -p'(B + \lambda I)p^* + \frac{1}{2}p'(B + \lambda I)p + \frac{1}{2}p^{*'}(B + \lambda I)p^* \\
&= \frac{1}{2}(p - p^*)'(B + \lambda I)(p - p^*)
\end{aligned}$$

The proof will be complete if we can show that $x'(B + \lambda I)x \geq 0$ for all $x \in \mathbb{R}^n$. The inequality will hold for all vectors proportional to $w = \pm(p - p^*)$ for some $p \in \partial D$, and this includes *all* vectors that are not orthogonal to p^* : Assume that $x'p^* \neq 0$. Then

$$p = p^* - \frac{2x'p^*}{\|x\|^2}x \in \partial D,$$

and

$$x = \frac{\|x\|^2}{2x'p^*}(p^* - p).$$

Finally, let $y'p^* = 0$. Then $y'(B + \lambda I)y = \lim_{\varepsilon \rightarrow 0} (y + \varepsilon p^*)'(B + \lambda I)(y + \varepsilon p^*)$. However, $(y + \varepsilon p^*)'(B + \lambda I)(y + \varepsilon p^*) \geq 0$ for all $\varepsilon \neq 0$, so in the limit, $y'(B + \lambda I)y \geq 0$!

May be you find a simpler proof?