Corrections for First and Second Printings

Partial Differential Equations: Methods and Applications
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Introduction

p. 7. Notice that $C_B(\Omega) \subset C(\Omega)$
For example, $C(\Omega, \mathbf{R}^N)$ denotes

1. First-Order Equations

There are some unfortunate inconsistencies in my discussion of noncharacteristic space curves and the solvability of the Cauchy problem that occurs on pages 15-16, 18, and 32-3. Corrections are given below.

p. 15 *Replace the first paragraph on the page by the following two paragraphs:*

In fact, it is also true that an integral surface S of the vector field $V = \langle a, b, c\rangle$ is *always* a union of characteristic curves: this can be proved using the uniqueness theorem for solutions of ordinary differential equations (see Exercise 1). Since the solutions of the characteristic equations are unique, we find that the integral surface S is unique. This means that we have proved the following.

Theorem. *If Γ is noncharacteristic, then the vector field $V = \langle a, b, c\rangle$ admits a unique integral surface S containing Γ.*

To find the solution $u(x, y)$ of (1), it remains only to replace the variables s and t by expressions involving x and y. Theoretically, this can be achieved exactly when the integral surface S is the graph of a function. We shall discuss this in more detail in the next two subsections.

p. 16 ...satisfied when Γ is noncharacteristic *and* γ has no stationary points.

Example 1. ...and we see that (9) holds: $2 \neq 0$.

p. 18 *Replace:* Since this geometric condition ...near Γ. *By:*

Therefore, we can solve the Cauchy problem provided Γ is noncharacteristic and $\gamma = (f(s), g(s))$ has no stationary points.

p. 23. Exercise 1. ...through P. (Assume the vector field V is C^1.)

Exercise 5. ...determine the values of x, y, and z for which it exists:

p. 26. *Displayed formula:* $u(x, y) =$

...for all values of (x, y).

p. 28. Exercise 4. Find a continuous weak solution.

p. 29. Exercise 8.

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_{max} & \text{for} \quad x < 0 \\ \rho_{max} & \text{for} \quad x > 0. \end{cases}$$

p. 32. ...(29) and (30) have been fixed, the integral surface S always exists ...

p. 33. *Replace the Theorem by:*

Theorem. *If Γ is noncharacteristic for (24) and functions ϕ, ψ exist satisfying (29) and (30), then there is an integral surface S containing Γ (which is unique for the choice of ϕ and ψ.)*

In order to obtain $u(x, y)$ from S, we need to have the Jacobian condition (8), just as in the quasilinear case.

p. 37.

$$(41) \qquad x^2 + y^2 = (1 \pm ct^2).$$

We may also study solutions of (34) . . . complete integral for (34)

p. 38-9. **Exercises 2 & 6.** Clairaut

2. Principles for Higher-Order Equations

p. 41. Differentiating these functions with respect . . .

. . . use the general solution discussed in Section 1.1.e.

p. 45. . . . and hence $u_{yyxx}(x,0) = 0 = u_{yxyx}(x,0) = \cdots$

. . . use to determine $u_{yyy}(x,0) = 1 + 2x - x^2$.

p. 46.

$$\partial_y^j \partial_x^k u = (i\partial_x)^j \partial_x^k u \qquad \text{for all } j,k \geq 0.$$

$$(4) \qquad u(x,y) \sim \cdots \frac{(i)^j g^{(j+k)}(0)}{j!\,k!}\, y^j z^k$$

$$(a+b)^m = \sum_{j+k=m} \frac{m!}{j!\,k!}\, b^j a^k$$

$$\sum_{m=0}^{\infty} \sum_{j+k=m} \frac{m!}{j!\,k!}\, b^j a^k = \sum_{m=0}^{\infty} (a+b)^m$$

p. 52. *Every instance of a ν should be η, and*

$$B = 2a\mu_x\eta_x + b(\mu_x\eta_y + \mu_y\eta_x) + 2c\mu_y\eta_y.$$

p. 54. **Example 8.** Consider $u_{xx} - u_{yy} = 0$ as in Example 1 . . .

p. 55. . . . all functions depend continuously on their variables.

p. 56.

> . . . *strictly hyperbolic* if the $N \times N$ matrix A has N distinct real eigenvalues . . . (19) is *hyperbolic* if A has all real eigenvalues . . . confirms that A has real eigenvalues and a basis of eigenvectors; . . .

p. 58. Nevertheless, a solution can be shown to exist (see Section 10.1).

p. 59. Exercise 3.

$$u(x,y) = \begin{cases} 0 & \text{if} \quad x \leq y \\ (x-y)^2 & \text{if} \quad x > y. \end{cases}$$

Exercise 10. . . . i.e. $W = \sqrt{4ac - b^2}$, . . . new coordinates μ, η transforming . . .

p. 62.

$$\int_{\mathbf{R}^2} \tilde{u}\, Lv\, d\mu d\eta = \int_{\mu>0} \mu^2\, v_{\mu\eta}\, d\mu d\eta = -\int_{\mu>0} 2\mu\, v_\eta\, d\mu d\eta = 0$$

$$(39) \qquad u_{\mu\eta} = d(\mu, \eta, u, u_\mu, u_\eta),$$

p. 63.

$$(41) \qquad \int_{\Omega^\pm} u^\pm L'v\, d\mu d\eta = \int_{\Omega^\pm} L(u^\pm) v\, d\mu d\eta + \int_{\partial\Omega^\pm} B^\pm(u^\pm, v)\, ds,$$

where ds is arclength and $B^{\pm}(u,v)$ is an expression ...

p. 65.

...Notice that we can easily take any number of distributional derivatives of $\delta(x)$; in fact, $D^\alpha\delta(x)$ is the object satisfying $\int D^\alpha\delta(x)\,v(x)\,dx = (-1)^{|\alpha|}\int \delta(x)D^\alpha v(x)\,dx = (-1)^{|\alpha|}D^\alpha v(0)$.

$$\int_{\mathbf{R}^n}\delta_\mu(x)v(x)\,dx = \int_{\mathbf{R}^n}\delta(x-\mu)\,v(x)\,dx = \int_{\mathbf{R}^n}\delta(y)\,v(y+\mu)\,dy = v(\mu).$$

...Therefore, a distribution should simply be thought of as a linear mapping $C_0^\infty(\mathbf{R}^n)\to\mathbf{R}$.

p. 66. *After (47):* ...if $f,g\in L^1_{\ell oc}(\mathbf{R}^n)$, ...

p. 67.

$$D^\alpha(F\star G) = (D^\alpha F)\star G = F\star D^\alpha G, \tag{52}$$

so derivatives of a convolution can be taken on either factor. In particular, if $f\in C^k(\mathbf{R}^n)$ and $g\in L^1_{\ell oc}(\mathbf{R}^n)$, one of which has compact support, then the lemma above implies $f\star g\in C^k(\mathbf{R}^n)$.

p. 69. Assuming that $f\in L^1(\mathbf{R}^n)$ has compact support, ...

p. 70.

1. (a) For complex-valued functions u and v on Ω, let $\langle u,v\rangle = \int_\Omega u\overline{v}dx$, where $\overline{v}$ denotes the complex conjugate of v. If the coefficients $a_\alpha(x)$ in (32a) are complex-valued functions, define the adjoint L^* so that $\langle Lu,v\rangle = \langle u,L^*v\rangle$ for all $u\in C^m(\Omega)$, $v\in C_0^m(\Omega)$.
   (b) For (complex) vector-valued functions $\vec{u}(x) = (u_1(x),\dots,u_N(x))$ on Ω, let $\langle\vec{u},\vec{v}\rangle = \int_\Omega(u_1\overline{v}_1+\cdots u_N\overline{v}_N)dx$. If the coefficients $a_\alpha(x)$ in (32a) are $N\times N$ matrix-valued functions, then (32a) defines a *system* of mth-order operators. Define the adjoint L^* so that $\langle L\vec{u},\vec{v}\rangle = \langle\vec{u},L^*\vec{v}\rangle$ for all $\vec{u}\in C^m(\Omega,\mathbf{R}^N)$, $\vec{v}\in C_0^m(\Omega,\mathbf{R}^N)$.

p. 70. Exercise 2. Let $Lu = u_{\mu\xi}$...(b)...with respect to $d\mu$ and then...

p. 71. Exercise 5. (b) For $\Omega = \mathbf{R}^n$, find an infinite sequence $\{\xi_k\}_{k=1}^\infty$ ...

Exercise 10. (b)

$$F(x) = \begin{cases} a^{-1}\sinh ax & \text{if} \quad x>0 \\ 0 & \text{if} \quad x<0. \end{cases}$$

p. 72. Exercise 13.

$$\begin{aligned} F_1(\xi,\eta) &= H(\xi)\,H(\eta), & F_2(\xi,\eta) &= -H(\xi)\,H(-\eta), \\ F_3(\xi,\eta) &= -H(-\xi)\,H(\eta), & F_4(\xi,\eta) &= H(-\xi)\,H(-\eta). \end{aligned}$$

3. The Wave Equation

p. 76. In fact, if $F(\mu)$ and...rectangle ABCD in the $\mu\eta$-plane...Using $u(\mu,\eta) = F(\mu)+G(\eta)$...

p. 82. ...a particular solution for a nonhomogeneous equation.

Exercise 3. Consider the initial/boundary ...

p. 89. ...(see Exercise 7) with $c_n = (n-2)(n-4)\cdots 1$:

$$
\begin{aligned}
u(x,t) =& \frac{1}{c_n\omega_n}\frac{\partial}{\partial t}\left(\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\xi|=1} g(x+ct\xi)\,dS_\xi\right) \\
&+ \frac{1}{c_n\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\xi|=1} h(x+ct\xi)\,dS_\xi.
\end{aligned}
\tag{42}
$$

p. 90. Exercise 2. . . . (a) by using (37), and . . .

Exercise 5.

$$
\begin{cases} v_{tt} = c^2\Delta v - m^2 v & \text{for } x \in \mathbf{R}^2 \text{ and } t > 0 \\ v(x,0) = g(x), & v_t(x,0) = h(x). \end{cases}
$$

p. 93. Notice that $\partial\Omega_\tau = C_\tau \cup (\overline{B}_0 \times \{0\}) \cup (\overline{B}_\tau \times \{\tau\})$. . . Moreover, the exterior unit normal ν on $\partial\Omega_\tau$ is given on . . .

p. 97.

$$
u_{\xi\eta} - \frac{\alpha+\beta}{4}u_\xi + \frac{\alpha-\beta}{4}u_\eta - \frac{\gamma}{4}u = 0.
$$

$$
w_{\xi\eta} + \frac{\lambda}{4}w = 0 \quad \text{where } \lambda = \frac{\alpha^2-\beta^2-4\gamma}{4}
\tag{62}
$$

4. The Laplace Equation

p. 100. . . . div$\vec{E}$ has a natural interpretation as . . .

p. 102.

The equation $X'' + n^2 X = 0$ has solutions $X_0 = c_0 t + d_0$ and $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ for $n = 1, \ldots$. This means that $u_0(r,\theta) = -c_0 \log r + d_0$ and $u_n(r,\theta) = (a_n \cos n\theta + b_n \sin n\theta)(c_n r^{-n} + d_n r^n)$ for $n = 1, \ldots$

p. 109. Taking c_2 so that $\bar{c}_2\omega_n = -1$, we find . . .

p. 110.

$$
u_\xi(x) = \int_{\mathbf{R}^3} \frac{\delta_\xi(y)}{|x-y|}\,dy = -4\pi\int_{\mathbf{R}^3} K(x,y)\delta_\xi(y)\,dy.
\tag{26}
$$

Now let us replace $\delta_\xi(y)$ by $\rho(y)$, . . .

$$
u(x) = \int_\Omega \frac{\rho(y)}{|x-y|}\,dy = -4\pi\int_\Omega K(x,y)\rho(y)\,dy
\tag{27}
$$

. . . compactly supported distribution $-4\pi\rho$; by the results of Section 2.3, u is a distribution solution of $\Delta u = -4\pi\rho$. . . . *expect* that $u \in C^2(\Omega)$ with $\Delta u = -4\pi\rho$ in Ω, . . .

$$
\int_{\partial\Omega} \frac{\rho(y)}{|x-y|}\,dS_y \quad \text{and} \quad \int_{\partial\Omega} \frac{\partial}{\partial\nu_y}(|x-y|^{-1})\mu(y)\,dS_y
\tag{28}
$$

. . . Poisson equations $\Delta u = -4\pi\rho\delta_{\partial\Omega}$ and $\Delta u = 4\pi\mu\frac{\partial}{\partial\nu}\delta_{\partial\Omega}$

p. 116. **Proof.** Taking $u \equiv 1$ in (33) shows . . .

p. 117. . . . and $\overline{B_a(0)} \subset \Omega$, then (44) implies . . .

p. 118.

$$\text{(49b)} \qquad \max_{\xi\in\Gamma} |D^\alpha u(\xi)| \leq \ \ldots$$

p. 119. If we let $M = \max\{|u(x)| : |x| = a\}$ ...

p. 120. Exercise 3. (c)... (In Section 8.2 we shall show ...

p. 129. ... will be found to be *convergence in square mean:*

p. 131.

$$\text{(76)} \qquad \ldots \quad B_n = \frac{b_n}{\sqrt{\lambda_n}} = \frac{1}{\sqrt{\lambda_n}} \int_\Omega h(x)\phi_n(x)\,dx.$$

5. The Heat Equation

p. 134. ... discuss the existence and uniqueness of solutions to these ...

p. 137. Exercise 3. Suppose $f(x,t) = \sum_{n=1}^\infty a_n(t)\phi_n(x)$ ...

p. 138. Exercise 5. ... initial temperature distribution $u(x,y,0) = g(x,y)$, ... heat diffusion $u(x,y,t)$.

p. 144.

It only remains to show that u is continuous. For s and t satisfying $0 \leq s < t$, let us define $w(x,t,s) = \int_{\mathbf{R}^n} K(x,y,t-s)\,f(y,s)\,dy$. We can apply Theorem 1 with $g(y) = f(y,s)$ to conclude that $w(x,t,s)$ is continuous in $x \in \mathbf{R}^n$ and $0 \leq s \leq t$. Therefore

$$u(x,t) = \int_0^t w(x,t,s)\,ds$$

p. 145. Exercise 4.

$$u(x,t) = \sum_{k=0}^\infty \frac{1}{(2k)!} x^{2k} \frac{d^k}{dt^k} e^{-1/t^2}$$

p. 146. Exercise 10. ... of the nonhomogeneous heat equation ...
Exercise 11. ... where g is continuous and bounded.

p. 147. We could start with $u \in C^{2;1}(U)$ ...
Now choose $\psi \in C^\infty(\mathbf{R}^{n+1})$ with $\psi \equiv 0$ on $B_\epsilon(0)$ and $\psi \equiv 1$ on $\mathbf{R}^{n+1}\backslash B_{2\epsilon}(0)$. ... But for $(x,t) \in B_\epsilon(\xi)$ and $(y,s) \in \operatorname{supp} v \subset B_{4\epsilon}(\xi) \cap (\mathbf{R}^{n+1}\backslash B_{3\epsilon}(\xi))$, we have ...

6. Linear Functional Analysis

p. 153. (ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ (where ...

p. 165. **Remarks.** ... the regularity of weak solutions in Chapter 8 using ...

p. 166.

$$\text{(17)} \qquad \int_\Omega (-\nu\nabla\vec{u} : \nabla\vec{v} + \vec{f}\cdot\vec{v})\,dx = -\int_\Omega p\,\mathrm{div}\vec{v}\,dx,$$

p. 168.

$$\text{(23)} \qquad B_L(u,v) = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \ - c(x)uv \right) dx$$

(and a similar summation sign is intended in the displayed equation after (25)).

p. 178. Since $\|\nabla u\|_p \leq C\|u\|_{1,p}$, we obtain...

p. 179.

$$\int_{\bar{B}_{t\sigma}} |\nabla u(\bar{z})|\, d\bar{z} \leq \left(\int_{\bar{B}_{t\sigma}} d\bar{z}\right)^{1/p'} \left(\int_{\bar{B}_{t\sigma}} |\nabla u(\bar{z})|^p\, d\bar{z}\right)^{1/p}$$

$$= \bar{\omega}_n^{1/p'} (t\sigma)^{n/p'} \left(\int_{\bar{B}_{t\sigma}} |\nabla u(\bar{z})|^p\, d\bar{z}\right)^{1/p},$$

and dominating the last term by $\|\nabla u\|_p$, we obtain

$$\left|\frac{1}{\bar{\omega}\sigma^n} \int_{B_\sigma} u(z)\, dz - u(x)\right| \leq 2\sigma^{1-\frac{n}{p}}\, \bar{\omega}_n^{-\frac{1}{p}} \left(\int_0^1 t^{-\frac{n}{p}}\, dt\right) \|\nabla u\|_p.$$

p. 180. ...where $C = 4\,\bar{\omega}_n^{-\frac{1}{p}} \int_0^1 t^{-n/p}\, dt < \infty$ since $n/p < 1$.

p. 182.

$$\|u\|_{3/2} \leq \frac{1}{3} \sum_{i=1}^{3} \|u_{x_i}\|_1 \leq \frac{1}{\sqrt{3}} \|\nabla u\|_1,$$

...moreover, we have explicitly found the constant $C(3,1) = 1/\sqrt{3}$.

$$\|\, |u|^r\, \|_{3/2} \leq \frac{r}{\sqrt{3}} \|\, |u|^{r-1} \nabla u\, \|_1.$$

(54) $$\|\, |u|^r\, \|_{3/2} \leq \frac{r}{\sqrt{3}} \|\, |u|^{r-1}\, \|_{p'}\, \|\nabla u\|_p$$

$$\left(\int |u|^{\frac{3p}{3-p}}\, dx\right)^{2/3} \leq \frac{2p}{\sqrt{3}(3-p)} \left(\int |u|^{\frac{3p}{3-p}}\, dx\right)^{\frac{p-1}{p}} \|\nabla u\|_p.$$

$$\|u\|_q \leq \frac{2p}{\sqrt{3}(3-p)} \|\nabla u\|_p \qquad \text{where} \quad q = \frac{3p}{3-p},$$

which is the Sobolev inequality with $C(3,p) = \frac{2p}{\sqrt{3}(3-p)}$.

p. 183. ...with the value $C(n,p) = \frac{p(n-1)}{\sqrt{n}(n-p)}$.

7. Differential Calculus Methods

p. 204. ... is a compact embedding by Corollary A ...

p. 207. Notice that the coercive functional of Example 1 satisfies this condition ...

p. 208. Exercise 8. (b) $(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$

p. 209. Exercise 10. Show that the coercive functional of Example 1 satisfies ....

p. 216. ... this is called the *maximin characterization* of the eigenvalues.

The monotonicity with respect to the domain does *not* apply to Neumann eigenvalues; see Exercise 6 for a counterexample. However, the maximin ...

p. 217. Exercise 6. Let Ω be the square $\{(x,y) : -1 < x, y < 1\}$ and $\hat{\Omega}$ be the slitted circle $\{(r,\theta) : 0 \le r < 1,\ -\pi < \theta < \pi\}$. Although $\hat{\Omega} \subset \Omega$, show that the Neumann eigenvalues do *not* satisfy $\mu_n(\hat{\Omega}) \ge \mu_n(\Omega)$ for all $n \ge 1$.

Exercise 8. $\Delta^2 u - \lambda u = 0$

p. 220. ... the *perturbation theory* of PDEs; i.e., if $F(0) = 0$ and the *linear* equation ...

p. 223.

$$F'(u)v = \operatorname{div}\left(\frac{\nabla v}{(1+|\nabla u|^2)^{1/2}} - \frac{(\nabla u \cdot \nabla v)\,\nabla u}{(1+|\nabla u|^2)^{3/2}}\right). \tag{54}$$

p. 224. Exercise 4. (b) Use $\mathcal{T}_z x - \mathcal{T}_z y = \int_0^1 \frac{d}{dt}\mathcal{T}_z(y+t(x-y))\,dt = \int_0^1 \mathcal{T}_z'(y+t(x-y))\,(x-y)\,dt$

8. Linear Elliptic Theory

p. 226.

$$\begin{aligned}\langle u,v\rangle_2 &= \int_{\mathbf{T}^n} \sum_{|\gamma|=2} D^\gamma u D^\gamma v\,dx + \langle u,v\rangle_1 \\ &= (2\pi)^n \sum_\alpha \left(\alpha_1^4 + \alpha_1^2\alpha_2^2 + \alpha_2^4 + \cdots + \alpha_n^4 + |\alpha|^2 + 1\right) u_\alpha v_{-\alpha} \\ \|u\|_2^2 &= (2\pi)^n \sum_\alpha (\alpha_1^4 + \alpha_1^2\alpha_2^2 + \alpha_2^4 + \cdots + \alpha_n^4 + |\alpha|^2 + 1)|u_\alpha|^2,\end{aligned} \tag{6a}$$

p. 232. Exercise 4. (a) Determine the eigenvalues λ for which...

p. 236.

$$\lambda^{-1} \int_\Omega \sum_{j,k}^n a_{jk} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}\,dx \tag{37}$$

p. 240. Exercise 1. $\|\nabla u\|_{0;\Omega_1}^2 \le C\left(\|\Delta u\|_{0;\Omega}^2 + \|u\|_{0;\Omega}^2\right)$

Exercise 5. ... together imply $\partial u/\partial x_j \in H^k(\Omega)$.

Exercise 8.

$$\|u\|_{2;\Omega}^2 \le C\left(\|\Delta u\|_{0;\Omega}^2 + \|u\|_{1;\Omega}^2\right) \quad \text{valid for all } u \in H^2(\Omega).$$

p. 241. ... apply the theorem to the inequality $L_0 u \ge -cu \ge 0$ which ...

p. 242.

$$\max_{x\in\overline{\Omega}} u(x) \le \max_{x\in\partial\Omega} u^+(x) \qquad \text{where } u^+(x) = \max(u(x), 0). \tag{49}$$

p. 243. *Delete:* notice that the maximum principle (applied to $-w$) shows $w(x) \ge N \ge 0$.

Now reads: ... how to construct $w(x)$. Now let $v = u - w$. The calculation ...

9. Two Additional Methods

p. 261.

$$\text{(7a)} \qquad \{\vec{u},\vec{w},\vec{v}\} = \int_\Omega (\vec{u}\cdot\nabla)\vec{w}\cdot\vec{v}\,dx \qquad \text{for } \vec{u},\vec{w},\vec{v}\in \tilde{H}_0^{1,2}(\Omega,\mathbf{R}^n).$$

p. 269. ...which is a consequence of $\int_{\mathbf{R}^n} K(x,y,t)\,dy = 1$, guarantees ...

p. 281. Exercise 6. (c) ... $\beta_0 = \lim_{t\to\infty}(\log\|S(t)\|)/t$.

11. Linear and Nonlinear Diffusion

p. 310. A calculation shows $Lv - v_t = \exp(-Ar^2)[4A^2(x-\xi)^2 - 2aA - 2bA(x-\xi) + c + 2A(t-\tau)] - c\exp(-A\epsilon^2)$. ... achieve its maximum on ∂B_δ, since $w(x_0,t_0) = \eta v(x_0,t_0) = 0$.

p. 312. 4. *Proof of (a).* ...such that $u(x_0,t) < M$ for $t < t_1$, ...
Corollary. ... *are nonnegative* ...

p. 319. *The proof of Theorem 4 should be replaced with the following.*

Proof. The *Gagliardo-Nirenberg inequality* generalizes (35) of Chapter 6:

$$\text{(18a)} \qquad \begin{gathered} \|u\|_{\ell,q} \le C\|u\|_{m,p}^{\theta}\|u\|_r^{1-\theta}, \quad \text{for all } u\in H^{2,2}(\Omega), \text{ provided} \\ \ell < m,\ \ell/m \le \theta \le 1, \text{ and } \ell - \frac{n}{q} < \theta(m - \frac{n}{p}) - (1-\theta)\frac{n}{r}; \end{gathered}$$

for a proof of (18a) see [Friedman] (where the desired estimate is even proved when $\ell - \frac{n}{q} \le \theta(m-\frac{n}{p}) - (1-\theta)\frac{n}{r}$, provided $m - \ell - n/p \notin \mathbf{N}$). Let $Y = H^{1,q}(\Omega)$, take $m = 2 = p = r$, and use the elliptic estimate $\|u\|_{2,2} \le C\|Au\|_2$ for $u \in D(A)$, to obtain

$$\text{(18b)} \qquad \|u\|_Y \le C\|Au\|_2^{\theta}\|u\|_2^{1-\theta}, \quad \text{for all } u\in D(A).$$

Using Young's inequality, (18b) implies the existence of K for which

$$\text{(18c)} \qquad \begin{gathered} \|u\|_Y \le \epsilon\,\|Au\|_2 + K\,\epsilon^{-\theta/(1-\theta)}\,\|u\|_2 \quad \text{for all } \epsilon > 0 \text{ and } u\in D(A), \\ \text{provided } 1/2 \le \theta \le 1 \text{ and } 1 - \frac{n}{q} < 2\theta - \frac{n}{2}. \end{gathered}$$

Another (more general) way of defining $A^{-\alpha}$ for $\alpha > 0$ is inspired by the Laplace transform:

$$\text{(19)} \qquad A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-sA}s^{\alpha-1}\,ds.$$

Using (19), we may show that $A^{-\alpha}: Y \to Y$ is bounded when $\theta < \alpha \le 1$; hence $X_\alpha \subset Y$ as claimed in (17a). In fact,

$$\|A^{-\alpha}u\|_Y \le \frac{1}{\Gamma(\alpha)}\int_0^\delta \|e^{-sA}s^{\alpha-1}u\|_Y\,ds + \int_\delta^\infty \|e^{-sA}s^{\alpha-1}u\|_Y\,ds.$$

Use (18c) with $\epsilon = s^{1-\theta}$ and (44) of Section 9.2 to estimate $\|e^{-sA}s^{\alpha-1}u\|_Y$ by $C\,s^{\alpha-\theta-1}\|u\|_2$, which is integrable near $s = 0$ since $\alpha - \theta > 0$; so the integral over $(0,\delta)$ is bounded by $C\|u\|_2$, which in turn is bounded by $C\|u\|_Y$ since $Y \subset L^2(\Omega)$. Since e^{-sA} is β-contractive with $\beta < 0$, the integral over (δ,∞) is bounded by $C\|u\|_Y$. Thus $\|A^{-\alpha}u\|_Y \le C\|u\|_Y$.

The proof of (17b) is similar; in fact, if $2\alpha > 1$ and $m = 0$, then (17a) may be used to obtain (17b). ♠

p. 324. For convenience, let us assume $\int_\Omega dx = 1$, and use the Poincaré inequality to conclude

$$V(t) = \ldots \qquad\qquad \ldots \geq (k-\epsilon)\|u\|_{1,2}^2 - C_\epsilon.$$

p. 325.

Taking $\epsilon = k/2$, we obtain

$$\|u\|_{1,2}^2 \leq \frac{2}{k}(V(t)+C). \tag{29}$$

...the desired bound (24) with $K(t) \equiv 2k^{-1}(V(0)+C)$, ...

p. 326. *Between (35) and (36):* Integrating, we obtain

13. Nonlinear Elliptic Equations

p. 364. Moreover, $G_w(\lambda_*, 0, 0)\colon w \to Lw + QH_u(\lambda_*, 0)w = Lw$, since $H(\lambda, u) = F(\lambda, u) - Lu$ implies $H_u(\lambda_*, 0) = F_u(\lambda_*, 0) - L = 0$.

p. 369. *After (29), delete* If we take ... *and substitute :*

Without loss of generality, we may assume $g \equiv 0$ in (13); this is achieved by the substitution $v = u - g$, where g has been extended to $\overline{\Omega}$. Then $\Delta v = f(x, v+g) + \Delta g \equiv \tilde{f}(x, v)$, where $\tilde{f}$ is again uniformly bounded.

With $g \equiv 0$ in (13), we may take $u_\pm \in C^\infty(\overline{\Omega})$ to be solutions of

$$\begin{cases} \Delta u = \mp M & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, $u_+ \geq 0 \geq u_-$, so that Theorem 1 implies that we have a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (13) with $g \equiv 0$. This yields the following result (for general g):

Theorem 2. *If $f(x,u)$ satisfies (16) and (29), and $g(x)$ satisfies (18), then there is a solution $u \in C^{2,\alpha}(\overline{\Omega})$ of (13).*

p. 370. *Before (32):* so that (13) with $g \equiv 0$ may be written

After (34):

A solution of (13) with $g \equiv 0$ and f satisfying (30) must satisfy the same equation with f replaced by f_0. But we may then apply Theorem 2. Since $f(x,u)$ satisfies (30) if and only if $\tilde{f}(x,u) \equiv f(x, u+g) + \Delta g$ does, we may extend this solvability to $g \not\equiv 0$:

Theorem 3. *Suppose $f(x,u)$ satisfies (16) and (30), and g satisfies (18). Then (13) admits a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

Hints and Solutions for Selected Exercises

Section 1.1

4. (a) $u(x,y) = \frac{1}{2}y^2 + x^2e^{-2y}$ exists for all x, y.
5. (a) $u(x,y,z) = h(xe^{-z}, ye^{-z})e^z$ exists for all x, y, z.
6. (a) $u(x,y) = (3y+1)^{1/3}$.
7. (a) $u = f((x+u)/(y+u))$, (b) $u = yf(x^2y + y^3 + u^2y)$.

Section 1.2

3. The shock curve is given by $x = \xi(y) = 1-(1+u_0y)^{1/2}$. The solution is $u(x,y) = 0$ for $x < \xi(y)$, and $u(x,y) = u_0(x-1)/(1+u_0y)$ for $x > \xi(y)$.
7. (b) Assuming a car has length ℓ, then $\rho(x,0) = 1/\ell$ for $|x - x_0| \leq \ell/2$, $\rho(x,0) = 0$ for $|x - x_0| > \ell/2$ describes a single car with midpoint at x_0. (One could also use a point-mass model for traffic flow, in which case $\rho(x,0) = \delta_{x_0}(x)$, using the delta function of Section 2.3.)

8. The shock curve is $x = \xi(t) = -\frac{1}{2}\,ct$. (See above correction for p. 29.)

Section 1.3

2. $u(x,y) = \pm y + \frac{1}{2}(1-x^2)$.
3. It is possible to solve the equation if $0 < a \le \frac{1}{4}$. If $0 < a < \frac{1}{4}$, then there are two solutions: $u(x,y) = (\sqrt{a}x \pm \frac{\sqrt{1-4a}}{2}y)^2$. But, for $a = \frac{1}{4}$, there is a unique solution $u(x,y) = \frac{1}{4}x^2$.

Section 2.1

5. The solution itself is a polynomal (which converges everywhere):

$$u(x,t) = \sum_{j,k=0}^{2j+k=n} \frac{(2j+k)!}{j!k!} a_{2j+k} t^j x^k.$$

Section 2.2

2. (a) $u(x,y) = f(y - x - \cos(x)) + g(y + x - \cos(x))$.
6. $u_1(x,t) = x + 5t$, $u_2(x,t) = -4t$.
7. (a) decouples, but (b) does not.
8. Let $\mu = x - \frac{t}{\sqrt{LC}}$, $\eta = x + \frac{t}{\sqrt{LC}}$, and $k = R/L$. Then $I(x,t) = \frac{1}{2}[I_0(\mu) + I_0(\eta) + \sqrt{\frac{C}{L}}V_0(\mu) - \sqrt{\frac{C}{L}}V_0(\eta)]e^{-kt}$, and $V(x,t) = \frac{1}{2}[\sqrt{\frac{L}{C}}I_0(\mu) - \sqrt{\frac{L}{C}}I_0(\eta) + V_0(\mu) + V_0(\eta)]e^{-kt}$.

Section 2.3

1. (a) $L^*v = \sum_{|\alpha|\le m}(-1)^{|\alpha|}\, D^\alpha(\overline{a}_\alpha v)$.
   (b) $L^*\vec{v} = \sum_{|\alpha|\le m}(-1)^{|\alpha|}\, D^\alpha(\overline{a}_\alpha^T \vec{v})$, where $\overline{a}_\alpha^T$ denotes the conjugate transpose matrix for a_α.
3. (a) ...a weak solution if $\int_\Omega f(\mu)\, v_\eta(\mu,\eta)\, d\mu\, d\eta = 0$ for all ... (c) No.
5. (b) ...extra assumptions on the coefficients a_k.
6. (a) ...Use this fact to conclude $|f * g(y) - f * g(x)| \le \epsilon \int |g(z)|\, dz$, whenever ...
   (b) Consider the distribution $G = \delta'$ on $\mathbf{R}^1$ and the function $f(x) = |x|$.
7. (b) $\langle a\partial_\nu \delta_\Gamma, v\rangle = -\int_\Gamma \nabla(a(z)v(z)) \cdot \nu(z)\, dz$.
8. Use the mean value theorem.
10. (a) Use e^{-ax} as an integrating factor.
11. (a) Apply the Lemma of this section to show that $u \in C(\mathbf{R})$. (b) Write $u(x) = \frac{1}{2}\int_{-\infty}^x (x-y)f(y)\, dy + \frac{1}{2}\int_x^\infty (y-x)f(y)\, dy$, and show $h^{-1}[u(x+h) - u(x)] \to \frac{1}{2}\int_{-\infty}^x f(y)\, dy - \frac{1}{2}\int_x^\infty f(y)\, dy$. Finally, use the hypotheses on f to verify $u' \in C(\mathbf{R})$. (c) Use (b) to compute u''.
12. (a) Use $u = u_p + u_h$ to find $u(x) = u_0 - \frac{1}{2}\int_0^\infty |y|f(y)\, dy + (u_0' + \frac{1}{2}\int_0^\infty f(y)\, dy)x + \frac{1}{2}\int_0^\infty |x-y|f(y)\, dy$.
13. (b) Use $v_{tt} - c^2 v_{xx} = -4c^2 v_{\mu\eta}$. One fundamental solution is $F_1(x,t) = (2c)^{-1}H(ct + x)H(ct - x)$, having support in $-ct < x < ct$ and $t > 0$. (See above correction for p. 72.)

Section 3.1

1. (a) $u(x,t) = x^3 + 3c^2xt^2 + c^{-1}\sin(x)\sin(ct)$.
2. The Fourier series solution is

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1-\cos(n\pi)}{n^2}\, \sin(nx)\, \sin(nt).$$

3. (a) The Fourier series solution is

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^\infty \frac{1}{k^2}\,[\cos(k\pi) - 1]\, \cos(kx)\, \sin(kt).$$

(b) Use the parallelogram rule for u_x in R_1 and R_2. The solution is *not* unique; it may be C^0, but it is *not* C^1.

4. Show that the solution given by d'Alembert's formula satisfies the boundary condition $u(0,t)=0$.
7. The solution is given by (9) with $L=\pi$. (a) $a_n(t)=c_n \sin\lambda_n t + d_n \cos\lambda_n t$ where $\lambda_n = (c^2n^2+m^2)^{1/2}$. (b) If $c^2<m^2$, then $a_1(t)$ is either trivial, or unbounded.

Section 3.2

1. (a) $|\alpha|^2 = c^{-2}$. (b) $\nabla g(x) = \alpha F'(\alpha\cdot x) = -\alpha h(x)$. (c) $u(x,t) = 1+x_1-x_2\pm c\sqrt{2}\,t$.
2. …and we can solve it by (39).
5. Let $u(x_1,x_2,x_3,t)=\cos(\frac{m}{c}x_3)\,v(x_1,x_2,t)$, and show u satisfies a three-dimensional wave equation with initial conditions. Compute u from (37), and then use this to compute v. (See above correction for p. 90.)
6. (b) No. This is a consequence of Huygen's principle, and may be proved by using (39).

Section 3.3

1. Use energy as a measure of the solution, and show a small change in the Cauchy data (on a compact set) results in only a small change at a later time.
2. Differentiate $\mathcal{E}_\Omega(t)$ with respect to t, and then use the boundary condition to show $\int_\Omega \sum u_{x_i} u_{x_i t}\,dx = -\int_\Omega u_t \Delta u\,dx$.
4. (a) The energy integral is $\mathcal{E}(t) = \int(|u_t|^2 + c^2|\nabla u|^2 + q(x)\,u^2)\,dx$, and may be used to define global and local energy. (b) Duplicate the proof of Theorem 2 for the energy defined above.

Section 3.4

5. $\pm\sqrt{k^2+\lambda}g'(x) = h(x)$.

Section 4.1

2. You should get a series of the form $a_o y + \sum_{k=1}^{\infty} a_k \cos(kx)\sinh(ky)$.
3. Let $w=u_1-u_2$ and use $\int_\Omega |\nabla w|^2\,dx = \int_{\partial\Omega} w\frac{\partial w}{\partial\nu}\,dS$.
4. (b) Consider $u(r,\theta) = r(a\cos\theta + b\sin\theta)$.
5. Use (9) with either Dirichlet or Neumann boundary condition to conclude $\int(qu^2 + |\nabla u|^2)\,dx = 0$.
8. Let $v(x) = \exp(-\alpha|x-x_1|^2) - \exp(-\alpha\epsilon^2)$ for $x\in B_\epsilon(x_1)$ where $\alpha>0$. For α sufficiently large, show that $\Delta v<0$ in $B_{\epsilon/2}(x_0)$. For $\eta>0$ sufficiently small, show $u+\eta v \le u(x_0)$ on $B_\epsilon(x_1)\cap B_{\epsilon/2}(x_0)$. Conclude that $\partial(u+\eta v)/\partial\nu \ge 0$ at x_0. Compute $\partial v/\partial\nu$ at x_0.

Section 4.2

7. Define $\tilde{u}(x_1,\cdots,x_n)$ by $\tilde{u}(x_1,\dots,x_n) = -u(x_1,\dots,-x_n)$ if $x_n\le 0$ and $\tilde{u}=u$ if $x_n\ge 0$. Show ….
12. Use $n^k = (1+\cdots 1)^k = \sum_{|\alpha|=k}(|\alpha|!/\alpha!)$ to show $|\alpha|! \le n^{|\alpha|}\alpha!$ for each $\alpha = (\alpha_1,\dots,\alpha_n)$.

Section 5.2

1. (a) Direct calculations. (b) and (c) Use the substitution $z=(x-y)/2\sqrt{t}$. (d) Write $u(x,t)-g(x) = \int K(x,y,t)(g(y)-g(x))\,dy$ and imitate the proof of Theorem 3 in Section 4.2.

Section 7.2

6. For $\hat{\Omega}$ you will encounter the half-integral Bessel function $J_{1/2}(z) = (\frac{2}{\pi z})^{1/2}\sin z$.

Section 7.3

7. Show $\langle F'(u)v,\phi\rangle = -\int_\Omega \frac{\nabla v\cdot\nabla\phi}{(1+|\nabla u|^2)^{1/2}}\,dx + \int_\Omega \frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla\phi)}{(1+|\nabla u|^2)^{3/2}}\,dx$.

Section 9.2

1. (b) $\dot{\mathcal{E}}(t) = \dot{y}\,\ddot{y} + ky\dot{y} = -c\dot{y}^2$.

(c) With $\mu = \sqrt{4k - c^2}/2$,

$$\mathbf{x}(t) = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = e^{-ct/2} \begin{pmatrix} \frac{c}{2\mu}\sin\mu t + \cos\mu t & \frac{1}{\mu}\sin\mu t \\ -\frac{k}{\mu}\sin\mu t & -\frac{c}{2\mu}\sin\mu t + \cos\mu t \end{pmatrix} \mathbf{x_0}$$

(d)

$$\mathbf{x}(t) = e^{-ct/2} \begin{pmatrix} 1 + \frac{c}{2}t & t \\ -kt & 1 - \frac{c}{2}t \end{pmatrix} \mathbf{x_0}$$

Section 11.3

2. . . . at least by $t = 2/\sqrt{\epsilon}$.

References

p. 412. [Kevorkian] J. Kevorkian, *Partial Differential Equations: Analytical Solution Techniques*, Wadsworth & Brooks/Cole, 1990.

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