# A WHITE NOISE APPROACH TO STOCHASTIC NEUMANN BOUNDARY VALUE PROBLEMS

HELGE HOLDEN AND BERNT ØKSENDAL

*Dedicated to Prof. Takeyuki Hida on the occasion of his 70th birthday*

ABSTRACT. We illustrate the use of white noise analysis in the solution of stochastic partial differential equations by solving explicitly the stochastic Neumann boundary value problem

$$LU(x) - c(x)U(x) = 0, \quad x \in \mathcal{D} \subset \mathbb{R}^d,$$
$$\gamma(x) \cdot \nabla U(x) = -W(x), \quad x \in \partial\mathcal{D}$$

where $L$ is a uniformly elliptic linear partial differential operator and $W(x)$, $x \in \mathbb{R}^d$, is $d$-parameter white noise.

## 1. INTRODUCTION

Since the seminal book by Hida [4] appeared in 1980, there has been a rapid development of white noise theory and its applications. In particular, white noise theory has found many spectacular applications in mathematical physics. See, e.g., [5, 7] and references therein.

In addition, white noise calculus turns out to be useful in the study of stochastic differential equations, both ordinary and partial. See [6].

The purpose of this paper is to illustrate the application of white noise calculus to stochastic partial differential equations (SPDEs) by studying the stochastic boundary value problem of Neumann type:

$$LU(x) - c(x)U(x) = 0 \text{ for } x \in \mathcal{D}, \tag{1.1}$$

$$\gamma(x) \cdot \nabla U(x) = -W(x) \text{ for } x \in \partial\mathcal{D}. \tag{1.2}$$

Here $\mathcal{D} \subset \mathbb{R}^d$ is a given bounded $C^2$ domain and $L$ is a partial differential operator of the form

$$L = \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.3}$$

$c(x) \geq c \geq 0$ and $\gamma(x)$ are given functions (satisfying certain conditions) and $W(x) = W(x,\omega)$, $\omega \in \Omega$, is *white noise.*

One may think of $U$ as a temperature in a medium governed by the differential equation $LU(x) - c(x)U(x) = 0$. Then the Neumann boundary condition models heat flux across the boundary which in our case is given by white noise. (See, e.g., [2].)

As pointed by Walsh [9] for similar SPDEs, it is not possible to find a solution $u(x,\omega)$ of (1.1)–(1.2) which is a regular stochastic process (random field), unless the dimension $d$ is very low. To cover the general case it is necessary to introduce some kind of weak solution concept. One possibility is to look for solutions $u(x,\omega)$ such that

$$u(\,\cdot\,,\omega) \text{ is a distribution (in } x\text{) for a.a. } \omega. \tag{1.4}$$

This is the approach chosen by Walsh.

For nonlinear SPDEs such an approach causes difficulties because one will have to define nonlinear operations on distributions. However, it is possible to adapt the Colombeau nonlinear theory of distributions to some nonlinear SPDEs. See, e.g., [8].

*Date*: February 22, 1999.

The other possibility is to look for solutions $u(x,\omega)$ such that

$$u(x,\,\cdot\,) \text{ is a } \textit{stochastic} \text{ distribution (in } \omega) \text{ for all } x. \tag{1.5}$$

Such an approach fits well with the white noise theory, where both the Hida space $(\mathcal{S})^*$ and the more general Kondratiev space $(\mathcal{S})$ of stochastic distributions are to our disposal. Moreover, in these spaces there is a natural product $\diamond$ (the Wick product) and a corresponding theory for nonlinear operations on distributions.

In this paper we will use this second approach and look for solutions of the type (1.5). We will prove that, under some conditions, the equation (1.1)–(1.2) has the unique solution

$$u(x,\omega) = \widehat{E}^x\left[\int_0^\infty \exp\big(-\int_0^t c(x_s(\hat\omega))ds\big)\; W(x_t(\hat\omega),\omega)\; d\xi_t(\hat\omega)\right] \tag{1.6}$$

where $(x_t,\xi_t) = (x_t(\hat\omega),\xi_t(\hat\omega))$, $\hat\omega \in \widehat{\Omega}$, is the solution of the Skorohod stochastic differential equation

$$dx_t = b(x_t)dt + \sigma(x_t)d\beta_t + \gamma(x_t)d\xi_t \tag{1.7}$$

where $\beta_t = \beta_t(\hat\omega)$ is the $d$-dimensional Brownian motion on a filtered probability space $(\widehat{\Omega},\widehat{\mathcal{F}}_t,\widehat{P}^x)$. The pair $(x_t,\xi_t)$ is unique under the conditions that $x_t \in \overline{\mathcal{D}}$ for all $t$, $\xi_t$ is a nondecreasing adapted process increasing only when $x \in \partial\mathcal{D}$. The process $\xi_t$ is called *local time* of $x_t$ at $\partial\mathcal{D}$ and the process $x_t$ is called the *reflection* (at $\partial\mathcal{D}$, at the angle $\gamma$) of the Ito diffusion $y_t$ given by

$$dy_t = b(y_t)\;dt + \sigma(y_t)\;d\beta_t. \tag{1.8}$$

For a more detailed explanation see Section 3. We refer to [1] and [3] for information about Skorohod stochastic differential equations and the associated Neumann boundary conditions.

The process $u(x,\,\cdot\,)$ defined in (1.6) belongs to the space $(\mathcal{S})^*$ for all $x$.

## 2. A brief review of the white noise theory

For the convenience of the reader we recall briefly the concepts and terminology we will use from white noise theory. For more information we refer to [6].

In the following we let $(\mu,\mathcal{S}'(\mathbb{R}^d),\mathcal{B})$ be the white noise probability space. Here $\Omega = \mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$, $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $\Omega$ and $\mu$ is defined by the property that

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = \exp(-\frac{1}{2}\|\phi\|^2) \tag{2.1}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ (the Schwarz space of rapidly decreasing smooth functions on $\mathbb{R}^d$), where $\langle\omega,\phi\rangle$ denotes the action of $\omega \in \mathcal{S}'(\mathbb{R}^d)$ on $\phi$ and $\|\phi\|^2 = \int_{\mathbb{R}^d}\phi(x)^2\;dx$, $dx$ being the Lebesgue measure on $\mathbb{R}^d$.

We let $\{\eta_k\}_{k=1}^\infty$ be the basis of $L^2(\mathbb{R}^d)$ consisting of tensor products of the *Hermite functions*

$$\xi_n(x) = \pi^{-1/4}\left((n-1)!\right)^{-1/2} e^{-x^2/2} h_{n-1}(\sqrt{2}\,x), \quad n \in \mathbb{N},\ x \in \mathbb{R} \tag{2.2}$$

where

$$h_m(y) = (-1)^m e^{y^2/2}\frac{d^m}{dy^m}\big(e^{-y^2/2}\big), \quad m = 0,1,2,\dots,\ y \in \mathbb{R} \tag{2.3}$$

are the *Hermite polynomials.*

The symbol $\mathcal{I}$ denotes the set of all multi-indices $\alpha = (\alpha_1,\dots,\alpha_n)$ where the $\alpha_i$'s are non-negative integers and $n \in \mathbb{N}$. According to the chaos expansion theorem any $f \in L^2(\mu)$ can be written uniquely

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha(\omega) \tag{2.4}$$

(convergence in $L^2(\mu)$) where $a_\alpha \in \mathbb{R}$ and

$$H_\alpha(\omega) = \prod_{j=1}^{n} h_{\alpha_j}\left(\langle \omega, \eta_j \rangle\right) \text{ if } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}. \tag{2.5}$$

For $-1 \le \rho \le 1$ and $q \in \mathbb{Z}$ we define the *Kondratiev norms* $\|\cdot\|_{\rho,q}$ by

$$\|F\|_{\rho,q}^2 = \sum_\alpha a_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{q\alpha} \tag{2.6}$$

if $F$ is the (formal) expansion

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha(\omega), \tag{2.7}$$

where $(2\mathbb{N})^{q\alpha} = \prod_{j=1}^n (2j)^{q\alpha_j}$ if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}$.

The corresponding *Kondratiev Hilbert spaces* are defined by

$$(\mathcal{S})_{\rho,q} = \{F : \|F\|_{\rho,q} < \infty\}. \tag{2.8}$$

The *Kondratiev (stochastic) test function spaces* are defined by (if $0 \le \rho \le 1$)

$$(\mathcal{S})_\rho = \bigcap_{r=1}^\infty (\mathcal{S})_{\rho,r}, \text{ with the projective topology.} \tag{2.9}$$

The *Kondratiev (stochastic) distribution* spaces are defined by (if $0 \le \rho \le 1$)

$$(\mathcal{S})_{-\rho} = \bigcup_{r=1}^\infty (\mathcal{S})_{-\rho,-r}, \text{ with the inductive topology.} \tag{2.10}$$

In particular, if we choose $\rho = 0$ in (2.9) we get the *Hida test function spaces*

$$(\mathcal{S}) = (\mathcal{S})_0 \tag{2.11}$$

and if we choose $\rho = 0$ in (2.10) we get the *Hida distribution spaces*

$$(\mathcal{S})^* = (\mathcal{S})_{-0}. \tag{2.12}$$

*The (singular, d-parameter) white noise* is defined by

$$W(x,\omega) = \sum_{k=1}^\infty \eta_k(x) H_{\epsilon^{(k)}}(\omega) \tag{2.13}$$

where $\epsilon^{(k)} = (0, 0, \dots, 1, \dots)$ with 1 on the $k$th place. We easily verify that $W(x,\cdot) \in (\mathcal{S})^*$ for all $x$ by considering

$$\begin{aligned} \|W(x,\cdot)\|_{0,-q}^2 &= \sum_k \eta_k^2(x) \left(\epsilon^{(k)}\right)! (2\mathbb{N})^{-q\epsilon_k} \\ &= \sum_k \eta_k^2(x)(2k)^{-q} < \infty \text{ for all } q > 1, \end{aligned}$$

because

$$\sup_k \Big( \sup_{x \in \mathbb{R}^d} |\eta_k(x)| \Big) < \infty.$$

Hence $W(x,\cdot) \in (\mathcal{S})_{0,-q}$ for all $q > 1$ and in particular $W(x,\cdot) \in (\mathcal{S})^*$.

To each $F(\omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha(\omega) \in (\mathcal{S})_{-1}$ we can associate the expansion

$$\mathcal{H}F(z) = \sum_{\alpha \in \mathcal{I}} a_\alpha z^\alpha \text{ for } z = (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}} \tag{2.14}$$

where $\mathbb{C}^{\mathbb{N}}$ is the set of all sequences $(z_1, z_2, \dots)$ of complex numbers $z_j$ and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots \text{ if } \alpha = (\alpha_1, \alpha_2, \dots)$$

The function $\mathcal{H}F(z)$ is called the *Hermite transform* of $F$. One can characterize the elements of $(\mathcal{S})_{-1}$ by means of their Hermite transforms as follows:

**Theorem 2.1.** *A formal expansion*

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha(\omega)$$

*belongs to* $(\mathcal{S})_{-1}$ *if and only if there exists* $q < \infty$ *such that*

$$\mathcal{H}F(z) = \sum_{\alpha \in \mathcal{I}} a_\alpha z^\alpha$$

*is a (uniformly convergent and) bounded analytic function on the infinite-dimensional ellipsoids*

$$\mathbb{K}_q(R) = \{(\zeta_1, \zeta_2, \dots) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2\}. \tag{2.15}$$

For a proof, see [6], Theorem 2.6.11.

The key to the solution of stochastic partial differential equations of the type (1.1)–(1.2) is the following, which is a special case of Theorem 4.1.1 in [6].

**Theorem 2.2.** *Suppose we seek a solution* $U(t,x) : G \to (\mathcal{S})_{-1}$ *of a linear stochastic partial differential equation of the form*

$$A(t, x, \partial_t, \nabla_x, U, W(x)) = 0. \tag{2.16}$$

*Suppose the Hermite transformed equation*

$$A(t, x, \partial_t, \nabla_x, u, \widetilde{W}(x,z)) = 0 \tag{2.17}$$

*has a (strong) solution* $u = u(t,x,z)$ *for each value of the parameter* $z \in \mathbb{K}_q(R)$ *for some* $q < \infty$, $R < \infty$. *Moreover, suppose that* $u(t,x,z)$ *and all its partial derivatives involved in* (2.17) *are uniformly bounded for* $(t,x,z) \in G \times \mathbb{K}_q(R)$, *continuous with respect to* $(t,x) \in G$ *for all* $z \in \mathbb{K}_q(R)$ *and analytic with respect to* $z \in \mathbb{K}_q(R)$ *for all* $(t,x) \in G$.

*Then there exists* $U(t,x) : G \to (\mathcal{S})_{-1}$ *such that* $u(t,x,z) = \mathcal{H}U(t,x,z)$ *for all* $(t,x,z) \in G \times \mathbb{K}_q(R)$ *and* $U(t,x)$ *solves (in the strong sense in* $(\mathcal{S})_{-1}$*) the equation* (2.16).

## 3. Solution of the stochastic Neumann boundary value problem

Before we state and prove the main result of this paper we briefly review some results about Skorohod stochastic differential equations and *deterministic* Neumann boundary value problems.

In the following we let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ with a $C^2$ boundary. This means that the boundary of $\mathcal{D}$, $\partial\mathcal{D}$, is locally the graph of a $C^2$ function (a twice continuously differentiable function). We assume that we are given Lipschitz continuous functions $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and a $C^2$ function $\gamma : \partial\mathcal{D} \to \mathbb{R}^d$ such that

$$\gamma(x) \cdot \nu(x) > 0 \text{ for all } x \in \partial\mathcal{D} \tag{3.1}$$

where $\nu(x)$ is the inward pointing unit normal at $x \in \partial\mathcal{D}$.

Consider, as in (1.7), the following *Skorohod* stochastic differential equation (in the unknown processes $x_t$, $\xi_t$)

$$dx_t = b(x_t)dt + \sigma(x_t)d\beta_t + \gamma(x_t)d\xi_t, \quad x_0 = x \in \overline{\mathcal{D}} \tag{3.2}$$

where we require that $x_t$, $\xi_t$ are adapted and

$$x_t \in \overline{\mathcal{D}} \text{ for all } t \geq 0. \tag{3.3}$$

$$\xi_t \text{ is continuous, nondecreasing and } \xi_t \text{ increases only when } x_t \in \partial\mathcal{D}. \tag{3.4}$$

The process $x_t$ is called the *reflection at* $\partial\mathcal{D}$ of the process $y_t$ given by

$$dy_t = b(y_t)dt + \sigma(y_t)d\beta_t, \quad y_0 = x \in \overline{\mathcal{D}} \tag{3.5}$$

and $\xi_t$ is called the *local time* of $x_t$ at $\partial\mathcal{D}$.

The following result can be found in [1], Theorem 12.1:

**Theorem 3.1.** *There exists a unique solution* $(x_t, \xi_t)$ *of the Skorohod stochastic differential equation* (3.2)-(3.4).

We proceed to consider the connection to Neumann boudary value problems: With $b$, $\sigma$ as above define

$$Lu(x) = \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{3.6}$$

where

$$a_{ij}(x) = \frac{1}{2}\big(\sigma\sigma^T\big)_{ij}(x) = \frac{1}{2}\sum_{k=1}^{n} \sigma_{ik}(x)\sigma_{kj}(x). \tag{3.7}$$

Assume that $L$ is *uniformly elliptic*, i.e., there exists an $\epsilon > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x) y_i y_j \geq \epsilon |y|^2 \tag{3.8}$$

for all $y = (y_1, \dots, y_n) \in \mathbb{R}^d$.

Let $f$ be a $C^2$ function on $\partial\mathcal{D}$ and let $c$ be a $C^2$ function on $\mathcal{D}$ such that there exists $\tilde{\epsilon} > 0$ with

$$c(x) \geq \tilde{\epsilon} \text{ for all } x \in \mathcal{D}. \tag{3.9}$$

The *Neumann problem* is to find a function $u \in C^2(\overline{\mathcal{D}})$ such that

$$Lu(x) - c(x)u(x) = 0 \text{ for } x \in \mathcal{D} \tag{3.10}$$

and

$$\nabla u(x) \cdot \gamma(x) = -f(x) \text{ for } x \in \partial\mathcal{D}. \tag{3.11}$$

Recall that if $h\colon G \subset \mathbb{R}^d \to \mathbb{R}$ is a given function and $0 < \lambda < 1$, we define the Hölder norms

$$\|h\|_{C^\lambda(G)} = \sup_{y \in G} |h(y)| + \sup_{y,z \in G} \frac{|h(y) - h(z)|}{|y - z|^\lambda} \tag{3.12}$$

and

$$\|h\|_{C^{2+\lambda}(G)} = \sum_{0 \leq |\alpha| \leq 2} \|\partial^\alpha h\|_{C^\lambda(G)}, \tag{3.13}$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq 2$ and

$$\partial^\alpha h = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_d}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} h.$$

The following result can be obtained by combining Theorem 6.2 with the conclusion of Section III.7 in [1] (see also [3]):

**Theorem 3.2.** *We have that*

(i) *Under the above assumptions there exists a unique solution* $u \in C^2(\overline{\mathcal{D}})$ *of the Neumann problem* (3.10)–(3.11).

(ii) *Moreover, the solution can be represented in the form*

$$u(x) = \widehat{E}^x \left[ \int_0^\infty \exp\big( - \int_0^t c(x_s) ds \big) f(x_t) d\xi_t \right], \tag{3.14}$$

*where* $(x_t, \xi_t)$, *with probability law* $\widehat{P}^x$, *is the solution of the Skorohod equation* (3.2)– (3.4) *and* $\widehat{E}^x$ *denotes expectation with respect to* $\widehat{P}^x$.

(iii) *For all* $\lambda \in (0,1)$ *there exists* $K < \infty$ *such that*

$$\|u\|_{C^{2+\lambda}(\overline{\mathcal{D}})} \leq K \, \|f\|_{C^\lambda(\partial\mathcal{D})}. \tag{3.15}$$

We can now state and prove the main result of this paper:

**Theorem 3.3.** *Let $\mathcal{D}$, $b$, $\sigma$, $\gamma$, $L$ and $c$ be as above. Then the stochastic Neumann problem*

$$LU(x) - c(x)U(x) = 0 \text{ for } x \in \mathcal{D}, \tag{3.16}$$

$$\gamma(x) \cdot \nabla U(x) = -W(x) \text{ for } x \in \partial\mathcal{D}, \tag{3.17}$$

*where $W(x) = W(x,\omega)$, $\omega \in \Omega$, is the $d$- parameter white noise, has a unique solution*

$$U : \overline{\mathcal{D}} \to (\mathcal{S})^*$$

*given by*

$$U(x) = U(x,\omega) = \widehat{E}^x \left[ \int_0^\infty \exp\big( - \int_0^t c(x_s) ds \big) W(x_t,\omega) d\xi_t \right], \tag{3.18}$$

*where $(x_t, \xi_t)$ solves the Skorohod equation* (3.2)–(3.4).

*Proof.* The key to our method is Theorem 2.2. So we consider the Hermite transformed equation

$$Lu(x,z) - c(x)u(x,z) = 0 \text{ for } x \in \mathcal{D}, \tag{3.19}$$

$$\gamma(x) \cdot \nabla u(x,z) = -\widetilde{W}(x,z) \text{ for } x \in \partial\mathcal{D} \tag{3.20}$$

where $z = (z_1, z_2, \dots) \in \mathbb{C}_0^{\mathbb{N}}$ (the set of finite sequences in $\mathbb{C}^{\mathbb{N}}$). As before the differential operators $L$ and $\nabla$ act on the $x$-variable and $z = (z_1, z_2, \dots) \in \mathbb{C}_0^{\mathbb{N}}$ is to be regarded as a parameter. Fix $z \in \mathbb{C}_0^{\mathbb{N}}$.

By considering separately the real and imaginary part of

$$\widetilde{W}(x,z) = \sum_{k=1}^\infty \eta_k(x) z_k$$

we get by Theorem 3.2 that the unique solution $u(x,z)$ of (3.19)–(3.20) is given by

$$u(x,z) = \widehat{E}^x \left[ \int_0^\infty \exp\big( - \int_0^t c(x_s) ds \big) \widetilde{W}(x_t,z) d\xi_t \right] \tag{3.21}$$

where $(x_t, \xi_t)$ solves the Skorohod equation (3.2)–(3.4). Moreover, $u(x,z)$ and all its partial derivatives up to order 2 are uniformly bounded for $(x,z) \in \overline{\mathcal{D}} \times \mathbb{K}_2(R)$ for all $R < \infty$. This follows by (3.15) plus the fact that

$$\begin{aligned}
\|\widetilde{W}(\,\cdot\,,z)\|_{C^\lambda(\partial\mathcal{D})} &\le \sup_k \|\eta_k\|_{C^\lambda(\partial\mathcal{D})} \sum_{k=1}^\infty |z_k| \\
&\le M_1 \sum_{k=1}^\infty (2k)^{-2} \sum_{k=1}^\infty |z_k|^2 (2k)^2 \\
&= M_2 \sum_{k=1}^\infty |z^{\epsilon^{(k)}}|^2 (2\mathbb{N})^{2\epsilon^{(k)}} \\
&\le M_2 \sum_{\alpha \ne 0} |z^\alpha|^2 (2\mathbb{N})^{2\alpha} \\
&< M_2 R^2 \quad \text{if } z \in \mathbb{K}_2(R)
\end{aligned}$$

and similar estimates for the partial derivatives. Similarly we see that $u(x,z)$ is analytic with respect to $z \in \mathbb{K}_2(R)$ for all $x \in \overline{\mathcal{D}}$. So by Theorem 2.2 there exists $U(x) \in (\mathcal{S})_{-1}$ such that

$$u(x,z) = \mathcal{H}U(x,z)$$

for all $(x,z) \in \overline{\mathcal{D}} \times \mathbb{K}_2(R)$ and $U(x)$ solves (in the strong sense) the stochastic Neumann equation (3.16)–(3.17).

Finally we verify that $U(x) \in (\mathcal{S})^*$: Since $U(x)$ has the expansion

$$U(x,\omega) = \sum_{k=1}^\infty g_k(x) H_{\epsilon^{(k)}}(\omega),$$

with

$$g_k(x) = \widehat{E}^x\left[\int_0^\infty \exp\big(-\int_0^t c(x_s)ds\big)\eta_k(x_t)d\xi_t\right]$$

we see that

$$\sup_k \left\{g_k^2(x)\epsilon^{(k)}!\,(2\mathbb{N})^{-q\epsilon^{(k)}}\right\} = \sup_k \left\{g_k^2(x)k^{-q}\right\} < \infty \text{ for all } q \geq 0.$$

□

## A remark about the solution

As pointed out in the introduction, our solution (3.18) is a *stochastic distribution.* This means that — just like for ordinary deterministic distributions — it acts on its corresponding test function space, which in this case is the space $(\mathcal{S})^*$ of Hida test functions. If we choose $\psi \in (\mathcal{S})^*$, then the action of $U$ on $\psi$ is given explicitly by

$$\langle U, \psi\rangle = \widehat{E}^x\left[\int_0^\infty \exp\big(-\int_0^t c(x_s)ds\big)\langle W(x_s,\cdot),\psi\rangle d\xi_s\right]$$

where

$$\begin{aligned}\langle W(x_s,\cdot),\psi\rangle &= \sum_{k=1}^\infty \eta_k(x_s)\langle H_{\epsilon^{(k)}},\psi\rangle = \sum_{k=1}^\infty \eta_k(x_s)\langle\langle\cdot,\eta_k\rangle,\psi\rangle \\ &= \sum_{k=1}^\infty \eta_k(x_s)E_\mu\left[\langle\omega,\eta_k\rangle\psi(\omega)\right].\end{aligned}$$

In particular, if $\psi(\omega) = \psi_l(\omega) = \langle\omega,\eta_l\rangle$ then

$$\langle U, \psi\rangle = \widehat{E}^x\left[\int_0^\infty \exp\big(-\int_0^t c(x_s)ds\big)\eta_l(x_s)d\xi_s\right].$$

We could regard this as the average of $U$ with respect to the (Gaussian) stochastic weight function $\psi_l(\omega)$.

## Concluding remarks

The main purpose of this paper has been to illustrate how white noise analysis can be used to solve stochastic partial differential equations by applying the method to stochastic Neumann boundary value problems of the type (1.1)–(1.2).

By inspecting the proof we see that the same method applies to the more general equations

$$\begin{aligned}LU(x) - c(x)U(x) &= -g(x) \text{ in } \mathcal{D} \\ \gamma(x)\cdot\nabla U(x) - \lambda(x)U(x) &= -f(x) \text{ on } \partial\mathcal{D}\end{aligned}$$

where $f\colon \partial\mathcal{D} \to (\mathcal{S})^*$ and $g\colon \mathcal{D} \to (\mathcal{S})^*$ are given $(\mathcal{S})^*$-valued functions and $\lambda(x) > 0$ is smooth. Moreover, the conditions we have assumed on $L$, $c$, $\gamma$, $\lambda$ and $\mathcal{D}$ can be relaxed. In fact, we do not even need to assume uniform ellipticity if we allow ourselves to consider $(\mathcal{S})^*$-valued solutions $U(x,\cdot)$ which solve the equation in a week sense with respect to $x$.

## References


[1] R. Bass, *Diffusions and Elliptic Operators*, Springer, Berlin 1998.
[2] G. F. Duff and D. Naylor, *Differential Equations of Applied Mathematics*, Wiley, New York, 1966.
[3] M. Freidlin, *Functional Integration and Partial Differential Equations*, Princeton University Press, Princeton 1985.
[4] T. Hida, *Brownian Motion*, Springer, Berlin 1980.
[5] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit, *White Noise Analysis*, Kluwer, Dordrecht 1993.
[6] H. Holden, B. Øksendal, J. Ubøe and T. Zhang, *Stochastic Partial Differential Equations*, Birkhäuser, Basel 1996.
[7] H.-H. Kuo, *White Noise Distribution Theory*, CRC Press, Boca Ratan 1996.

[8] M. Oberguggenberger and F. Russo, Nonlinear SPDEs: Colombeau solutions and pathwise limits. In: *Stochastic Analysis and Related Topics VI*, (L. Decreusefond, J. Gjerde, B. Øksendal, and A. S. Üstünel, eds.), Birkhauser, Boston, 1998, pp. 319–332.

[9] J. B. Walsh, An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint-Flour XIV 1984*, (R. Carmona, H. Kesten, and J. B. Walsh, eds.), Lecture Notes in Mathematics, vol. 1180, Springer, Berlin, 1984, pp. 236–433.

Department of Mathematical Sciences, Norwegian University of Science and Technology, N–7491 Trondheim, Norway
*E-mail address*: holden@math.ntnu.no
*URL*: http://www.math.ntnu.no/~holden/

Department of Mathematics, University of Oslo, N–0316 Oslo, Norway, and Norwegian School of Economics and Business Administration, Hellevn 30, N–5035 Bergen–Sandviken, Norway
*E-mail address*: oksendal@math.uio.no