INTERNAL DISTANCES AND JOHN DOMAINS

OLE JACOB BROCH

Abstract. In this note it is proved that a bounded, simply connected subdomain D of $\mathbb{R}^2$ is a John domain if and only if there exists a quasidisk Ω and a homeomorphism $F\colon \Omega \to D$ such that F is bilipschitz with respect to internal length distances in Ω and D. It is also shown that all metrics that are locally Euclidean and quasisymmetrically equivalent to the length internal distance in a John disk must in fact be equivalent to the internal distance in D. Together these results explain precisely why the internal length distance is the right one for John domains.

1. Introduction

The purpose of this note is to point out some simple but important consequences of theory developed by Bonk, Heinonen and Rohde in [3].

Let D be a bounded subdomain of $\mathbb{R}^2$. D is a *c-John domain*, where $c \geq 1$ is a constant depending only on D, if for every pair of points z_1, z_2 in D there is a rectifiable arc β joining z_1 and z_2 in D with

$$\min\{\ell(\,\beta[z_1, z]\,), \ell(\,\beta[z, z_2]\,)\} \leq c\,\operatorname{dist}(z, \partial D) \text{ for all } z \in \beta.$$

Here $\ell(\alpha)$ denots the length of some arc α, and $\alpha[x, y]$ is the subarc of α from x to y, for any x and y in α. A simply connected planar John domain is often called a *John disk*. See Näkki and Väisälä [16]. The *internal length distance* between z_1 and z_2 in some domain D is given by

$$\lambda_D(z_1, z_2) = \inf\{\ell(\alpha)\colon \alpha \text{ rectifiable arc from } z_1 \text{ to } z_2 \text{ in } D\}.$$

John disks are *one sided* quasidisks. By this people often mean that John disks have a lot of the same geometric properties as quasidisks, but with Euclidean distances replaced by internal distances. See e.g. Näkki and Väisälä [16], Kim and Langmeyer [15], Broch [5, 6, 7]. Recall that a *K-quasidisk* is the image of the unit disk $\mathbb{D}$ under some global K-quasiconformal map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. I will prove the following result, which explains why quasidisks and John disks are basically the same thing when one uses internal distances.

Date: June 13, 2006.

2000 *Mathematics Subject Classification.* Primary 30C62, Secondary 30C20.

Key words and phrases. John disk, internal distance, bilipschitz map.

Theorem 1. *A bounded, simply connected domain D in $\mathbb{R}^2$ is a c-John disk if and only if there exists a K-quasidisk Ω and an L-bilipschitz homeomorphism $F\colon (\Omega, \lambda_\Omega) \to (D, \lambda_D)$. The constants involved depend only on each other.*

By (D, λ_D) I mean the metric space consisting of the domain D with the metric λ_D.

The proof of Theorem 1 is based on an important theorem due to Bonk, Heinonen and Rohde [3]. The proof of this result, in turn, is based on a technique by S. Rohde [18]. We give the proof of Theorem 1 in Section 2. Some consequences of Theorem 1 will be presented in Section 3.

The internal length metric is not the only metric suitable for John disks. The *internal diameter distance*, defined by

$$\delta_D(z_1, z_2) = \inf\{\operatorname{diam}(\alpha)\colon \alpha \text{ arc from } z_1 \text{ to } z_2 \text{ in } D\},$$

is often used. It is sometimes more convenient to use the δ-metric because one does not have to take into account the possibility that there are boundary points that are not rectifiably accessible. It is well known, however, that $\delta_D \leq \lambda_D \leq M\,\delta_D$ in a c-John disk, with $M = M(c)$, i.e. δ_D and λ_D are equivalent. See for instance Ghamsari-Näkki-Väisälä [13] and Broch [7]. In Section 4 I will prove the following.

Theorem 2. *Let $D \subset \mathbb{R}^2$ be a John disk (with constant c). Let d be a metric in D such that d is locally Euclidean and $d(z_1, z_2) \geq |z_1 - z_2|$. If d is quasisymmetrically equivalent to λ_D, then d is (bilipschitz) equivalent to λ_D.*

By a *locally Euclidean metric* in a domain $D \subset \mathbb{R}^n$ we will mean a metric d on D such that for every z in D there is a ball $B \subset D$ in which we have $d(z_1, z_2) = |z_1 - z_2|$. For all locally euclidean metrics d we clearly have that $d \leq \lambda_D$. Both λ_D and δ_D are locally Euclidean. Two metrics d_1 and d_2 on D are *quasisymmetrically equivalent* if the map $\mathrm{id}\colon (D, d_1) \to (D, d_2)$ is quasisymmetric. The metrics d_1 and d_2 are *equivalent* if $\mathrm{id}\colon (D, d_1) \to (D, d_2)$ is bilipschitz.

Given two metric spaces (X, d_X) and (Y, d_Y) and a homeomorphism $f\colon X \to Y$, f is called *η-quasisymmetric* if there exists a strictly increasing homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ such that

$$\frac{d_Y(h(x_1), h(x_2))}{d_Y(h(x_1), h(x_3))} \leq \eta\left(\frac{d_X(x_1, x_2)}{d_X(x_1, x_3)}\right)$$

for all $x_1, x_2, x_3 \in X$, $x_1 \neq x_3$. A map f between metric spaces (X, d_X) and (Y, d_Y) is *L-bilipschitz* if

$$L^{-1}\, d_X(x, y) \leq d_Y(f(x), f(y)) \leq L\, d_X(x, y).$$

The map f is *locally L-bilipschitz* if every $x \in X$ has a neighbourhood in which f is L-bilipschitz. A homeomorphism of a domain in $\mathbb{R}^n$ onto another is bilipschitz with

respect to internal distances if and only if it is locally bilipschitz with respect to the Euclidean distance.

I will write (D, d) for the metric space consisting of a domain D with the metric D. Euclidean disks are denoted by $B(z, r)$. $\mathbb{T}$ denotes the boundary of the unit disk $\mathbb{D}$.

2. Proof of Theorem 1

Let J_f denote the Jacobian of a quasiconformal map, defined almost everywhere. The following is a local version of an observation made by S. Rohde in [18]. The proof is essentially the same as in [18], Lemma 2.1.

Lemma 1 ([18], Lemma 2.1)**.** *Suppose that D and D' are simply connected domains in $\mathbb{R}^2$, and that we have orientation preserving quasiconformal maps $g\colon \mathbb{D} \to D$ and $f\colon \mathbb{D} \to D'$ satisfying $L^{-1} J_f(z) \le J_g(z) \le L\, J_f(z)$ almost everywhere. Then $F = f \circ g^{-1}\colon D \to D'$ is locally bilipschitz or, equivalently, bilipschitz with respect to internal distances.*

With every positive continuous function ϱ on $\mathbb{D}$, called a *density* ([3]), we can associate a metric space $\mathbb{D}_\varrho = (\mathbb{D}, d_\varrho)$, where the metric d_ϱ is defined by

$$d_\varrho(z_1, z_2) = \inf_\alpha \int_\alpha \varrho(z)\, |dz|. \tag{1}$$

The infimum is taken over all rectifiable curves α joining z_1 and z_2 in $\mathbb{D}$. When f is conformal in $\mathbb{D}$, we write $\varrho_f(z) = |f'(z)|$. We write $B_\varrho(z, r) = \{z \in \mathbb{D}\colon d_\varrho(a, z) < r\}$ for a ball in the metric d_ϱ. The following elementary observation is useful.

Lemma 2. *If $f\colon \mathbb{D} \to D$ is a conformal map, then $d_{\varrho_f}(z_1, z_2) = \lambda_D(f(z_1), f(z_2))$.*

Following Bonk, Koskela and Rohde [4], we say that ϱ is a *conformal density* if there are constants A_1 and A_2 such that the following conditions hold. Firstly,

$$A_1^{-1}\, \varrho(z_2) \le \varrho(z_1) \le A_1\, \varrho(z_2) \tag{2}$$

for every $z_1, z_2 \in B(\, z, (1 - |z|)/2\,)$, $z \in \mathbb{D}$. Secondly, let $\mu_\varrho(E) = \int_{E \cap \mathbb{D}} |\varrho(z)|^2\, dm$ (where m is two dimensional Lebesgue measure) for every Borel set $E \subset \mathbb{R}^2$; then ϱ is to satisfy

$$\mu_\varrho(B_\varrho(a, r)) \le A_2\, r^2. \tag{3}$$

For example, if f is conformal in $\mathbb{D}$, then $\varrho(z) = |f'(z)|$ satisfies conditions (2) and (3) in $\mathbb{D}$. See [4], Section 2.2 for the details. A density ϱ is a *doubling conformal density* (see [3]) if it satisfies (2), (3) and the following doubling condition

$$\mu_\varrho(B(x, 2r)) \le A_3\, \mu_\varrho(B(x, r)),$$

with $x \in \mathbb{D}$, $r > 0$, and where $A_3 \ge 1$ is a constant. Euclidean disks are used here.

We will use the following version of Theorem 1.15 in [3], which relates doubling conformal densities to globally defined quasiconformal maps.

Lemma 3 ([3], Theorem 1.15)**.** *A positive continuous function ϱ on $\mathbb{D}$ is a doubling conformal density if and only if there exists a quasiconformal map $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a constant $A \geq 1$ such that $A^{-1}\varrho(z) \leq J_g(z)^{1/2} \leq A\,\varrho(z)$ for almost every $z \in \mathbb{D}$.*

Lemma 4. *Suppose that $D \subset \mathbb{R}^2$ is a bounded, simply connected domain, and let $f\colon \mathbb{D} \to D$ be a conformal map. Then $\varrho_f(z) = |f'(z)|$ is a doubling conformal density on $\mathbb{D}$ if and only if D is a John disk.*

Proof. By Lemma 2 the map $f\colon (\mathbb{D}, d_{\varrho_f}) \to (D, \lambda_D)$ is an isometry. Then $f\colon \mathbb{D} \to (D, \lambda_D)$ is η-quasisymmetric if and only if $\mathrm{id}\colon \mathbb{D} \to (\mathbb{D}, d_{\varrho_f})$ is η-quasisymmetric. Thus by Proposition 2.11 in [3] $\varrho_f(z) = |f'(z)|$ is doubling if and only if $f\colon \mathbb{D} \to (D, \lambda_D)$ is quasisymmetric. By Section 7 in [16] (see also Väisälä [21]) this happens if and only if D is a John disk. □

Proof of Theorem 1. For the necessity, let D be a c-John disk and let $f\colon \mathbb{D} \to D$ be a conformal map. By Lemma 4 $\varrho_f(z) = |f'(z)|$ is a doubling conformal density in $\mathbb{D}$. By Lemma 3 there is a constant A, depending only on ϱ_f (hence only on c), and a $K = K(c)$-quasiconformal map $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$A^{-1}\varrho_f(z) \leq J_g(z)^{1/2} \leq A\,\varrho_f(z) \text{ a.e.}$$

Let $\Omega = g(\mathbb{D})$. Then by definition Ω is a K-quasidisk. Because f is conformal, $\varrho_f(z) = |f'(z)| = J_f^{1/2}$ and we have that

$$A^{-2}J_f(z) \leq J_g(z) \leq A^2\,J_f(z) \text{ a.e.}$$

By Lemma 1 the map $F = f \circ g^{-1}$ is locally L-bilipschitz, $L = L(c)$. Since F is a homeomorphism, and thus in particular injective, we have that

$$L^{-1}\lambda_\Omega(z_1, z_2) \leq \lambda_D(F(z_1), F(z_2)) \leq L\,\lambda_\Omega(z_1, z_2).$$

The sufficiency is not hard to prove from first principles. One can also appeal to the fact that quasidisks are John disks, and to Theorem 3.6 in [16], which says that the John condition is preserved under maps that are quasisymmetric with respect to internal distances. □

3. Some consequences of Theorem 1

Next we give some examples to illustrate the usefulness of Theorem 1 when proving necessary conditions for John disks. The idea is as follows: A necessary condition for quasidisks involving Euclidean distances translates, by Theorem 1, to a necessary condition for John disks with internal distances.

3.1. **Snowflakes, dimension and John disks.** Let $\mathcal{S}$ denote Rohde's class of *snowflake-like* curves, as constructed in Section 3 of [18]. The classic von Koch snowflake is a prototype of a snowflake-like curve. Rohde proved in [18], Theorem 1.1, that every quasicircle is a (Euclidean) bilipschitz image of some curve S in $\mathcal{S}$. Thus combining Rohde's result with Theorem 1, we immediately obtain the following.

Corollary 1. *A bounded, simply connected domain $D \subset \mathbb{R}^2$ is a John disk if and only if there are $S \in \mathcal{S}$ and a bilipschitz map $F\colon (\Omega, \lambda_\Omega) \to (D, \lambda_D)$, where Ω is the bounded component of $\mathbb{R}^2 \setminus S$.*

Thus the collection of bounded domains bounded by some $S \in \mathcal{S}$ contains *all* bounded John disks up to application of a locally bilipschitz map.

If (X, d) is a metric space and $B \subset X$ is a subset, let $\dim_A(B, d)$ and $\dim_H(B, d)$ denote the Assouad and Hausdorff dimensions of B, respectively. (Assouad dimension is also known as the uniform metric dimension.) For the definitions, see Heinonen [14], Chapter 10. We will omit d when we use the Euclidean standard metric.

Corollary 2. *Let D be a bounded Jordan c-John domain. Then*

$$\dim_A(\partial D, \lambda_D) = \dim_A \partial\Omega, \quad \dim_H(\partial D, \lambda_D) = \dim_H \partial\Omega$$

where Ω is the quasidisk in Theorem 1. In particular,

$$\dim_H(\partial D, \lambda_D) \le \dim_A(\partial D, \lambda_D) \le a < 2,$$

where $a = a(c)$.

Proof. The Assouad dimension of a K-quasicircle is bounded from above by a constant $a < 2$ depending only on K. See e. g. [18], Lemma 4.1. Furthermore, the Assouad dimension is invariant under bilipschitz maps. The inequality $\dim_H(\partial D, \lambda_D) \le \dim_A(\partial D, \lambda_D)$ always holds. □

Note that all snowflake-like curves are boundaries of John disks, [18]. Hence $(\partial D, \lambda_D)$ may have Hausdorff dimension arbitrarily close to 2. On the other hand it is not difficult to prove that, for any Jordan domain $D \subset \mathbb{R}^2$,

$$\dim_H \partial D \le \dim_H(\partial D, \lambda_D).$$

Using this with Corollary 2 we obtain the following (well known) result. See e.g. [4], Section 7 for much more general results.

Corollary 3. *Let D be a bounded Jordan c-John domain. Then there is a constant $a = a(c) < 2$ such that $\dim_H \partial D \le a < 2$.*

3.2. **Two extension results.** The following is a one-sided version of a result due to Gehring for quasidisks ([10], Theorem 2.11).

Corollary 4 ([7], Theorem 4.7)**.** *Let $D \subset \mathbb{R}^2$ be bounded Jordan c-John domain. Suppose that $f\colon (\partial D, \lambda_D) \to (\partial D', \lambda_{D'})$ is an L-bilipschitz map. Then f has an extension $\tilde{f}\colon (\overline{D}, \lambda_D) \to (\overline{D'}, \lambda_{D'})$ which is also bilipschitz*

Proof. If D is a Jordan John domain and $f\colon (\partial D, \lambda_D) \to (\partial D', \lambda_{D'})$ is a bilipschitz map, then D' is a John domain too. See Broch [7], Theorem 4.7. Thus there are quasidisks Ω and Ω' and bilipschitz maps $F\colon (\Omega, \lambda_\Omega) \to (D, \lambda_D)$, $G\colon (\Omega', \lambda_{\Omega'}) \to (D', \lambda_{D'})$ by Theorem 1. Recalling that internal distances are comparable to Euclidean distances in quasidisks, we see that f induces a bilipschitz map $\tilde{F}\colon \partial\Omega \to \partial\Omega'$, because all bilipschitz maps may be extended as a bilipschitz map to the boundary. By [10], Theorem 2.11, $\tilde{F}$ has an extension as a bilipschitz map $\tilde{F}\colon \overline{\Omega} \to \overline{\Omega'}$. Then $\tilde{f} = G \circ \tilde{F} \circ F^{-1}\colon D \to D'$ is the desired extension. □

Note that in [7] the same result is proved also for unbounded John disks. It can be asked whether the proof given in the present note is simpler than the one in [7], since the tools developed in [18] and [3] are quite deep and powerful.

The second extension result may be known to many, but I have not seen it stated explicitly before. It appeared in my thesis [6] with a more involved proof. Again, this is a one sided result for John disks; the original version appears in, e.g., [18], (2.1). See also [3], Lemma 3.9.

Corollary 5. *Let D be a bounded Jordan domain, and let $f\colon \mathbb{T} \to (\partial D, \lambda_D)$ be an η-quasisymmetric homeomorphism. Then D is a John disk. Furthermore, f has an extension $f\colon \overline{\mathbb{D}} \to \overline{D}$ with the following properties.*

i) *$f\colon \overline{\mathbb{D}} \to (\overline{D}, \lambda_D)$ is ν-quasisymmetric.*
ii) *$f|_{\mathbb{D}}\colon \mathbb{D} \to D$ is K-bilipschitz with respect to hyperbolic distances.*
iii) *If $z \in \mathbb{D}$ and $J \subset \mathbb{T}$ is an arc with*

$$L^{-1} \leq \operatorname{dist}(z, \mathbb{T})/\ell(J) \leq \operatorname{dist}(z, J)/\ell(J) \leq L$$

we have

$$M^{-1}\ell(J)|D_f(z)| \leq \operatorname{diam} f(J) \leq M\ell(J)|D_f(z)|.$$

Here ν, η and the constants involved depend only on each other.

Proof. That D is a John disk follows from [7], Lemma 4.6. Let f, J etc. be as in the statement of Corollary 5, and let $F\colon \Omega \to (D, \lambda_D)$ be the L-bilipschitz map from Theorem 1. Now $\tilde{F} = F^{-1} \circ f\colon \mathbb{T} \to \partial\Omega$ is a β-quasisymmetric map onto a quasicircle. Hence it has an extension, as in [18] (2.1), with the properties stated in Corollary 5. It is not difficult to check, using the fact that F is bilipschitz with respect to internal and hyperbolic distances, that $f = F \circ \tilde{F}\colon \mathbb{D} \to D$ is an extension of the kind we are looking for. □

3.3. **A bound for hyperbolic distance.** Let

$$j_D(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)}\right),$$

$$j'_D(z_1, z_2) = \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right).$$

Denote hyperbolic distance in a hyperbolic domain D by $h_D(z_1, z_2)$.

Corollary 6 ([15], Theorem 4.1)**.** *Let D be a c-John disk. Then there is a constant b depending only on c such that $h_D(z_1, z_2) \leq b\, j'_D(z_1, z_2)$.*

Proof. Let Ω be the K-quasidisk associated with D via Theorem 1, and let $F\colon \Omega \to D$ be the corresponding L-bilipschitz map; F may be assumed to be L-bilipschitz both with respect to internal and hyperbolic distances. Take ζ_1, ζ_2 in Ω with $f(\zeta_i) = z_i$. There is a constant a depending only on K, and hence only on c, such that $h_\Omega(\zeta_1, \zeta_2) \leq a j_\Omega(\zeta_1, \zeta_2)$; see Gehring-Osgood [12]. Then

$$\begin{aligned} h_D(z_1, z_2) &\leq L\, h_\Omega(\zeta_1, \zeta_2) \\ &\leq aL \log\left(1 + \frac{|\zeta_1 - \zeta_2|}{\operatorname{dist}(\zeta_1, \partial\Omega)}\right)\left(1 + \frac{|\zeta_1 - \zeta_2|}{\operatorname{dist}(\zeta_2, \partial\Omega)}\right) \\ &\leq aL \log\left(1 + L^2\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + L^2\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right) \\ &\leq aL^3\, j'_D(z_1, z_2). \end{aligned}$$

Here we used that $\operatorname{dist}(z, \partial D)$ is the same in the Euclidean and the λ_D metric. □

Of course this result follows from the more general Lemma 2.13 in Bonk, Heinonen and Koskela [2]. My point is simply that in view of Theorem 1, results about John disks follow "trivially" from results about quasidisks.

4. Proof of Theorem 2

Proof of Theorem 2. First recall that in a John disk one may always use hyperbolic segments γ as the curves in the definition. See Gehring-Hag-Martio [11] or Näkki and Väisälä [16]. Secondly, remember the Gehring-Hayman theorem that says that if $\gamma[z_1, z_2]$ is the hyperbolic geodesic joining z_1 and z_2 in D, then

$$\ell(\gamma[z_1, z_2]) \leq K\lambda_D(z_1, z_2) \tag{4}$$

for a universal constant K, the Gehring-Hayman constant. See e.g. Pommerenke [17], Theorem 4.20 or [4], Theorem 3.1.

Take z_1 and z_2 in D. Since D is a John disk we have that

$$\ell(\gamma[z_1, z_0]) = \ell(\gamma[z_2, z_0]) \leq c\, \operatorname{dist}(z_0, \partial D), \tag{5}$$

where z_0 is the mid-point - with respect to (Euclidean) arc length - of the hyperbolic geodesic joining z_1 and z_2 in D. By the quasisymmetric equivalence of d and λ_D we have

$$\frac{d(z_1, z_2)}{d(z_1, z_0)} \leq \eta\left(\frac{\lambda_D(z_1, z_2)}{\lambda_D(z_1, z_0)}\right) \leq \eta\left(K\frac{\ell(\gamma[z_1, z_2])}{\ell(\gamma[z_1, z_0])}\right) = \eta(2K) = a$$

where K is the Gehring-Hayman constant. Arguing similarly we obtain

$$1/a \leq d(z_1, z_2)/d(z_1, z_0) \leq a. \tag{6}$$

If $|z_1 - z_0| < \operatorname{dist}(z_0, \partial D)$, we have $d(z_1, z_0) \geq |z_1 - z_0| = \lambda_D(z_1, z_0)$. Then using (6) and (4) it follows that

$$d(z_1, z_2) \geq \frac{d(z_1, z_0)}{a} \geq \frac{\lambda_D(z_1, z_0)}{a} \geq \frac{\ell(\gamma[z_1, z_0])}{Ka} = \frac{\ell(\gamma[z_1, z_2])}{2Ka} \geq \frac{\lambda_D(z_1, z_2)}{2Ka}.$$

If $|z_1 - z_0| \geq \operatorname{dist}(z_0, \partial D)$ we have by (6) and (5)

$$d(z_1, z_2) \geq \frac{d(z_1, z_0)}{a} \geq \frac{|z_1 - z_0|}{a} \geq \frac{\operatorname{dist}(z_0, \partial D)}{a} \geq \frac{\ell(\gamma[z_1, z_0])}{ac} \geq \frac{\lambda_D(z_1, z_2)}{2ac}.$$

□

5. Remarks

Remark 1. i) In general the constant L in Theorem 1 cannot be too small. Because Ω is a quasidisk, there is a constant $\mathcal{L} = \mathcal{L}(\Omega) > 1$ such that whenever $1 \leq L < \mathcal{L}$ and F is locally L-bilipschitz, then $F(\Omega)$ is a quasidisk. See Theorem 8 in Gehring [8] for the details.

ii) We cannot expect Ω in Theorem 1 to be the unit disk, for in such cases D would have to be a so called *internal chord-arc domain*. See Väisälä [20, 21]. On the other hand Bishop [1], Corollary 1.5, has proved that every simply connected domain in $\mathbb{R}^2$ may be mapped onto the unit disk by a map that is lipschitz with respect to internal distances.

iii) Refering Theorem 1 again, it would be interesting to know if it is always possible to find a *conformal* map $f\colon \Omega \to D$ which is (internally) bilipschitz.

Remark 2. In Corollaries 2 to 5 I make the assumption that D be a *Jordan domain*, i.e. that ∂D is a Jordan curve. This is done only in order to make statements and technicalities simpler, not because it is essential. If $\overline{(D, \lambda_D)}$ denotes the completion of the metric space (D, λ_D), then $\partial_\lambda D = \overline{(D, \lambda_D)} \setminus (D, \lambda_D)$ has the structure of a Jordan curve. This is well explained in e. g. [4], Section 6 and [3], Section 2.

Remark 3. i) The conclusion in Theorem 2 holds in inner uniform domains (in $\mathbb{R}^n$) in which the Gehring-Hayman theorem holds for quasihyperbolic geodesics. The proof is the same.

ii) A way to state Theorem 2 is that whenever a locally Euclidean metric is in the conformal gauge of the internal length distance in a John disk, it is also in its metric gauge. See Heinonen [14, Chapter 15] for the terminology used here.

iii) The assumption that $d(z_1, z_2) \geq |z_1 - z_2|$ in the formulation of Theorem 2 is non-trivial in the sense that a locally Euclidean metric does not have to be greater than Euclidean distance *globally*. Note that with the assumptions in Theorem 2 we will always have $d(z_1, z_2) = |z_1 - z_2|$ in any convex subdomain of D.

References

[1] C. J. Bishop. *Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture.* Ark. Mat. **40** (2002), 1–26.

[2] M. Bonk, J. Heinonen, and P. Koskela. *Uniformizing Gromov hyperbolic spaces.* Astérisque **270** (2001).

[3] M. Bonk, J. Heinonen, and S. Rohde. *Doubling conformal densities.* J. reine angew. Math. **541** (2001), 117–141.

[4] M. Bonk, P. Koskela, and S. Rohde. *Conformal metrics on the unit ball in Euclidean space.* Proc. London Math. Soc. **77** (1998), 635–664.

[5] O. J. Broch. *Quadrilaterals and John disks.* Comput. Methods Funct. Theory. **4** (2004), 419–434.

[6] O. J. Broch. *Geometry of John disks.* Dr. scient.-thesis, Doctoral Theses at NTNU 2005:21, NTNU, 2005.

[7] O. J. Broch. *Extension of internally bilipschitz maps in John disks.* Ann. Acad. Sci. Fenn. Math. **31** (2006), 13–30.

[8] F. W. Gehring. *Injectivity of local quasi-isometries.* Comment. Math. Helv. **57** (1982), 202–220.

[9] F. W. Gehring. *Characteristic Properties of Quasidisks.* Les presses de l'université de Montréal (1982).

[10] F. W. Gehring. *Extension of Quasiisometric Embeddings of Jordan Curves.* Complex Var. Theory Appl. **5** (1986), 245–263.

[11] F. W. Gehring, K. Hag, and O. Martio. *Quasihyperbolic geodesics in John domains.* Math. Scand. **65** (1989), 75–92.

[12] F. W. Gehring and B. Osgood. *Uniform Domains and the Quasi-Hyperbolic Metric.* J. Anal. Math. **36** (1979), 50–74.

[13] M. Ghamsari, R. Näkki, and J. Väisälä. *John disks and Extension of Maps.* Monatsh. Math. **117** (1994), 63–94.

[14] J. Heinonen. *Lectures on Analysis on Metric Spaces.* Springer 2001.

[15] K. Kim and N. Langmeyer. *Harmonic measure and hyperbolic distance in John disks.* Math. Scand. **83** (1998), 283–299.

[16] R. Näkki and J. Väisälä. *John disks.* Expo. Math. **9** (1991), 3–43.

[17] Ch. Pommerenke. *Boundary Behaviour of Conformal Maps.* Springer-Verlag, 1992.

[18] S. Rohde. *Quasicircles modulo bilipschitz maps.* Rev. Mat. Iberoam. **17** (2001), 643–659.

[19] J. Väisälä. *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Matematics, **229**. Springer 1971.

[20] J. Väisälä. *Homeomorphisms of bounded length distortion.* Ann. Acad. Sci. Fenn. Ser. A. I. Math. **12** (1987), 303–312.

[21] J. Väisälä. *Quasiconformal maps of cylindrical domains.* Acta Math. **162** (1989), 201–225.

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail address: olejacb@math.ntnu.no