BLOCH FUNCTIONS AND QUASIDISKS

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Abstract. The following observation is made: A simply connected proper subdomain $D \subset \mathbb{R}^2$ is a K-quasidisk if and only if there exists a constant $b > 0$ such that when f is a Bloch function and c is a constant, $c > \|f\|_{\mathcal{B}(D)}/b$, then $f(z) = c \log g'(z)$ for a conformal map g in D. There is an analogous result for harmonic Bloch functions due to Astala and Gehring, and an extension result for harmonic Bloch functions in a quasidisk is commented on.

1. Introduction

Let D be a simply connected domain of hyperbolic type in the plane $\mathbb{R}^2$. A function f analytic in D is a *Bloch function*, or lies in the *Bloch space* $\mathcal{B}(D)$, if

$$\|f\|_{\mathcal{B}(D)} = \sup_{z \in D} |f'(z)| \operatorname{dist}(z, \partial D) < \infty. \tag{1.1}$$

For example, if g is univalent in D, then the function $f(z) = \log g'(z)$ lies in $\mathcal{B}(D)$ (see 2.2 below). Thus $\mathcal{B}(D)$ is non-empty for every D. On the other hand, every Bloch function f in the unit disk $\mathbb{D} = \{z \in \mathbb{R}^2 \colon |z| < 1\}$ may be written on the form

$$f(z) = c \log g'(z), \tag{1.2}$$

where g is conformal in $\mathbb{D}$, and c is some constant; [5, Theorem 9.4].

It is thus natural to ask: given a simply connected doman D and an analytic function $f \in \mathcal{B}(D)$, is it always possible to find some constant c and a conformal map g in D such that $f(z) = c \log g'(z)$?

The short answer is no, which is seen by considering an easy example in $\mathbb{D}\backslash(-1, 0]$. The complete answer to the question, however, is that every Bloch function may be written on the form (1.2) if and only if D is a quasidisk; see Theorem 2.3. There is an analogous result for real valued harmonic Bloch functions, originally due to Astala and Gehring [1]. See (2.6) and Theorem 2.7. In Section 2 we will also see that real valued harmonic functions in quasidisks, with "small enough" BMO (or, equivalently, Bloch) norms, extend to $\mathbb{R}^2$ as the logarithm of the Jacobian of a quasiconformal map (Corollary 2.10).

2. Representation of Bloch functions in the plane

By a K-quasidisk I mean the image of a disk or a half plane under a K-quasiconformal self map of $\overline{\mathbb{R}^2}$.

I will assume throughout this section that D is a simply connected proper subdomain of $\mathbb{R}^2$. Then $\varrho_D(z)$ will denote the hyperbolic metric in D, chosen so that

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$\varrho_{\mathbb{D}}(z) = 2(1-|z|^2)^{-1}$. It follows from Schwarz's lemma and Koebe's theorem (see e. g. [6, Corollary 1.4]) that

$$\frac{1}{2\,\mathrm{dist}(z,\partial D)} \leq \varrho_D(z) \leq \frac{2}{\mathrm{dist}(z,\partial D)}, \tag{2.1}$$

where $\mathrm{dist}(z,\partial D)$ denotes the distance from the point z to the boundary ∂D of D.

2.2. If f is a function analytic and locally univalent (i.e. $f' \neq 0$) in D, then

$$T_f = \frac{f''}{f'}$$

is the *logarithmic* or *pre-Schwarzian* derivative of f. Let $\tau(D)$ be the supremum of all constants $a > 0$ such that f is injective whenever f is analytic and locally univalent in D with

$$\sup_{z\in D} |T_f(z)|\varrho_D(z)^{-1} \leq a.$$

Note that $a = 0$ implies that f is an affine map, hence univalent. Then $\tau(D) > 0$ if and only if D is a K-quasidisk; [1], Theorems 1.3A and 1.4A, and [3], Theorem 4.24. $\tau(D)$ depends only on K and vice versa; $\tau(D) \leq 1/2$ for all domains D, [8].

On the other hand we have $\sup_{z\in D} |T_g(z)|\varrho_D(z)^{-1} \leq 4$ for every conformal map g in any simply connected domain D, and this bound is sharp; [4, Theorem 1]. Thus by (2.1) we have $\|\log g'\|_{\mathcal{B}(D)} \leq 8$ for every conformal map g in D.

Using the injectivity criteria of Martio-Sarvas [3] and Astala-Gehring [1] just mentioned, we may characterize quasidisks as follows.

Theorem 2.3. *A simply connected proper subdomain D of $\mathbb{R}^2$ is a K-quasidisk if and only if there exists a constant $b > 0$ such that for every $f \in \mathcal{B}(D)$ and every $c > \|f\|_{\mathcal{B}(D)}/b$ there exists a conformal map g defined in D with*

$$f(z) = c \log g'(z).$$

The constants K and b depend only on each other.

In particular if $\|f\|_{\mathcal{B}(D)} < b$ we may take $c = 1$.

Proof. Our proof is similar to that of Corollary 4.6 in [1]. It also has some similarities to those of [5, Theorem 9.4] and [6, Proposition 4.1].

Suppose first that D is a K-quasidisk. Then $\tau(D) > 0$ and this number depends only on K by [3, Theorem 4.24]. Let f be in $\mathcal{B}(D)$ and let $c > 2\|f\|_{\mathcal{B}(D)}/\tau(D)$. Now define a function analytic and locally univalent in D as follows. Fix z_0 in D and let

$$g(z) = \int_{z_0}^{z} \exp\left(\frac{f(\zeta)}{c}\right) d\zeta,$$

for every z in D, so that $f(z) = c \log g'(z)$. Then we have

$$\begin{aligned} |T_g(z)|\varrho_D(z)^{-1} &\leq 2|T_g(z)|\,\mathrm{dist}(z,\partial D) \\ &= \frac{2}{c}|f'(z)|\,\mathrm{dist}(z,\partial D) \leq \frac{2}{c}\|f\|_{\mathcal{B}(D)} < \tau(D), \end{aligned} \tag{2.4}$$

and g is univalent in D. We see that we may take $b = \tau(D)/2$.

Conversely, suppose that D has the following property. There exists a constant $b > 0$ such that if $f \in \mathcal{B}(D)$ and $c > \|f\|_{\mathcal{B}(D)}/b$, then there is a conformal map g in

D with $f(z) = c \log g'(z)$. Let h be any function analytic and locally univalent in D, and assume that $\sup_{z\in D} |T_h(z)|\varrho_D^{-1}(z) \leq b/2$. If we let

$$f(z) = \log h'(z),$$

then

$$\|f\|_{\mathcal{B}(D)} = \sup_{z\in D} |T_h(z)| \operatorname{dist}(z, \partial D) < b.$$

Thus $1 > \|f\|_{\mathcal{B}(D)}/b$, and by assumption there is a g conformal in D such that

$$f(z) = \log g'(z).$$

This means that $\log g'(z) = \log h'(z)$, and $h(z) = g(z) + a$. Because g is univalent this implies that h is univalent. Hence h is univalent whenever $|T_h(z)|\varrho_D(z)^{-1} < b/2$, and $\tau(D) \geq b/2 > 0$. Then D is a K-quasidisk with $K = K(b)$ by [1, Theorem 1.4A]. □

I believe it is of some interest to point out the following.

Corollary 2.5. *Let $D \subset \mathbb{R}^2$ be a K-quasidisk and let f be a Bloch function with $\|f\|_{\mathcal{B}} \leq c < \tau(D)/2$. Then $f = \log g'$, where g is a conformal map in D with a quasiconformal extension to $\overline{\mathbb{R}^2}$. In particular $g(D)$ is a K'-quasidisk with $K' = K'(K, c)$.*

Proof. Since $\|f\|_{\mathcal{B}(D)}/b < 1$ we have $f(z) = \log g'$ by Theorem 2.3. Besides

$$\begin{aligned}\sup_{z\in D} |T_g(z)|\varrho_D(z)^{-1} &\leq 2 \sup_{z\in D} |T_g(z)| \operatorname{dist}(z, \partial D)\\ &\leq 2|f'(z)| \operatorname{dist}(z, \partial D)\\ &\leq 2c < \tau(D),\end{aligned}$$

and the claim follows by Theorem 1.5A in [1]. □

We now turn to harmonic Bloch functions. If $D \subset \mathbb{R}^n$ is a domain, we say that a harmonic function $u\colon D \to \mathbb{R}$ lies in the *harmonic Bloch space* $\mathcal{B}_h(D)$ if

$$\|u\|_{\mathcal{B}(D)} = \sup_{x\in D} |\nabla u| \operatorname{dist}(x, \partial D) < \infty. \tag{2.6}$$

Note that I wrote $\|\cdot\|_{\mathcal{B}(D)}$ also for the seminorm in $\mathcal{B}(D)$, trusting that it will not cause confusion.

According to Astala and Gehring (proof of Cor. 4.6 in [1] and (2.9) below) every harmonic function in a simply connected domain D of the form $u = \log J_f$, where f is conformal in D, is a Bloch function. J_g denotes the Jacobian of a quasiconformal map g. As an analogue to Theorem 2.3 we note that the converse - i.e. that every harmonic Bloch function may be written as $u = c \log J_f$ - holds if and only if D is a quasidisk. This result is due to Astala and Gehring [1, Cor. 4.9], but not exactly in the wording given here. Recall the definition of $\tau(D)$ from 2.2.

Theorem 2.7 ([1], Corollary 4.9)**.** *Let D be a simply connected proper subdomain of $\mathbb{R}^2$. Then D is a K-quasidisk if and only if there is a constant $b > 0$ such that for every $u \in \mathcal{B}_h(D)$ and every $c > \|u\|_{\mathcal{B}(D)}/b$ there is a map f conformal in D with*

$$u(z) = c \log J_f(z).$$

If D is a quasidisk we can take $b = \tau(D)/2$; in the converse case $\tau(D) \geq b/2$.

The analogy with analytic functions goes even further. The following is an extension result in the spirit of Lemma 2.5 and Theorem 1.5A in [1].

Lemma 2.8. *Let D be a K-quasidisk. Then there is a constant $a = a(K)$ such that if u is harmonic with $\|u\|_{\mathcal{B}(D)} \leq c < a$, then $u(z) = \log J_g(z)$ where g is a conformal map in D with a $K' = K'(c, K)$-quasiconformal extension to $\overline{\mathbb{R}^2}$. In particular $g(D)$ is a quasidisk too.*

Proof. Let $a = b/2$, where b is the constant in Theorem 2.7. If $\|u\|_{\mathcal{B}(D)} \leq c < a = b/2$, we see from Theorem 2.7 that we may write $u(z) = \log J_g(z)$ for g conformal in D. Furthermore we have

$$\begin{aligned}\|u\|_{\mathcal{B}(D)} &= \sup_{z\in D} |\nabla u(z)| \operatorname{dist}(z, \partial D)\\ &= 2 \sup_{z\in D} |\nabla \log |g'(z)|| \operatorname{dist}(z, \partial D)\\ &= 2 \sup_{z\in D} |T_g(z)| \operatorname{dist}(z, \partial D)\\ &\leq 2c < b = \tau(D).\end{aligned}$$

Thus it follows from Theorem 1.5A in [1] that g has a K'-quasiconformal extension to $\mathbb{R}^2$ (with $K' = K'(K, c)$) and $g(D)$ is a quasidisk. □

Lemmas 4.1 and 4.2 in [1] give the following relations between the Bloch seminorm and the BMO-norm $\|u\|_{*,D}$ of harmonic functions:

$$\|u\|_{*,D} \leq \|u\|_{\mathcal{B}(D)} \leq 12\|u\|_{*,D}. \tag{2.9}$$

Thus every harmonic function in BMO is a Bloch function and conversely. A real valued, locally integrable function v defined in a domain $D \subset \mathbb{R}^2$ is of *bounded mean oscillation*, or lies in BMO(D), if

$$\|v\|_{*,D} = \sup_B \frac{1}{m(B)} \int_B |v - v_B| \, dm < \infty,$$

where the supremum is taken over all disks B with $\overline{B} \subset D$, and m is planar Lebesgue measure. v_B is the average of v over B.

Corollary 4.9 in [1] actually states that a simply connected domain D is a quasidisk if and only if every harmonic function u in D with $\|u\|_{*,D} \leq c$ can be written on the form $u(z) = \log J_f(z)$ for a conformal map f. By way of (2.9) one can also see that there is a formulation of Lemma 2.8 with BMO-norms.

A celebrated result due to Jones [2, Theorem 2] states that all BMO-functions u in a quasidisk D has BMO-extensions to $\mathbb{R}^2$ (and conversely that all domains with the BMO-extension property are quasidisks). We know slightly more if u is in addition harmonic.

Corollary 2.10. *Suppose that D is a K-quasidisk. Then there is a constant $a = a(K)$ such that every real valued harmonic function with $\|u\|_{*,D} \leq c < a$ has an extension $v \in \mathrm{BMO}(\mathbb{R}^2)$ with $v = \log J_g$, for some K'-quasiconformal map g of $\mathbb{R}^2$; $K' = K'(c, K)$ and $\|v\|_{*,D} \leq a' = a'(c, K)$.*

Proof. Let $a = b/24$, where b is the constant in Theorem 2.7, $b = b(K)$. If $\|u\|_{*,D} \leq c < a$, then by (2.9) we have $\|u\|_{\mathcal{B}(D)} \leq 12\|u\|_{*,D} < b/2$. By Lemma 2.8 $u = \log J_f$ for an f conformal in D, and furthermore f has a $K' = K'(c, K)$-quasiconformal extension g to $\mathbb{R}^2$. Then $v = \log J_g$ extends $\log J_f$. By Theorem 1 in [7] the logarithm of the Jacobian of a quasiconformal map $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a BMO-function.

By Lemmas 3 and 5 in [7] $\|v\|_{*,\mathbb{R}^2} \leq a' = a'(c, K)$. Thus $v = \log J_g$ is an extension of u as claimed. □

Remark 2.11. The representations of Bloch functions presented above are of course not unique. If, however, $u = \log J_f = \log J_g$ for conformal maps f and g, then $|f'| = |g'|$, which means that $f = a\,g + c$, where a and c are constants, $|a| = 1$. Likewise, if the (analytic) Bloch function f may be written as $f = \log g' = \log h'$, then $g = h + c$ for some constant c.

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