On reflections in Jordan curves

Ole Jacob Broch

olejacb@math.ntnu.no

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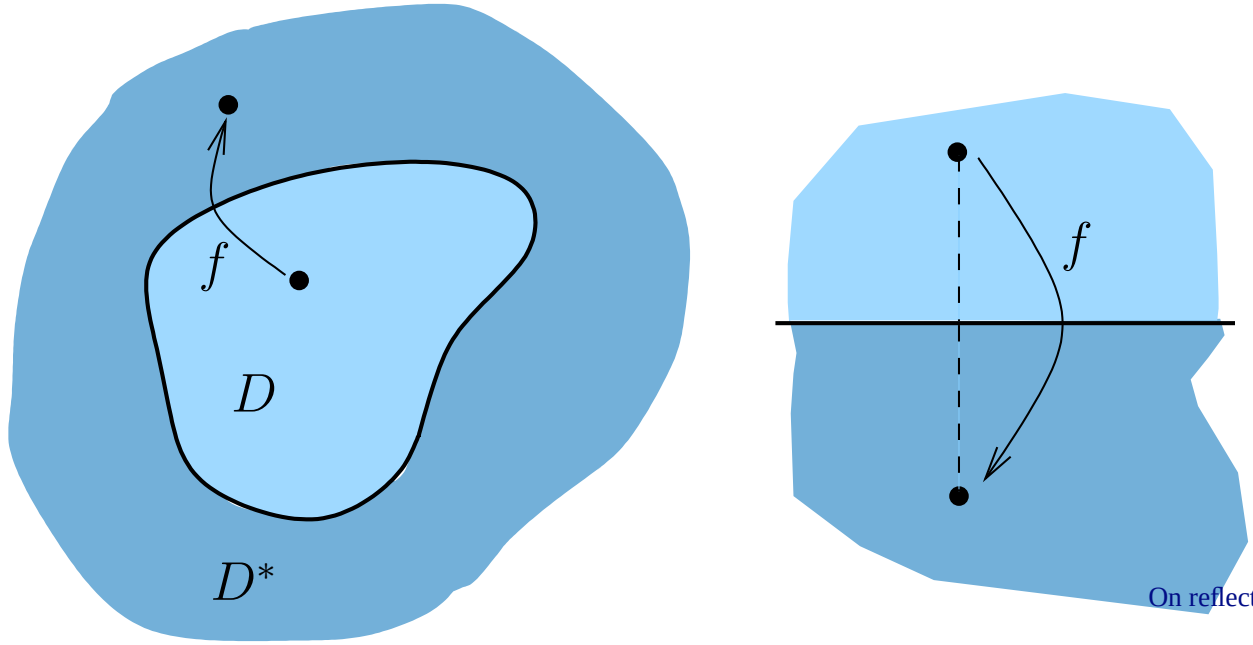
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Gehring and Hag (2001): A domain $D \subset \overline{\mathbb{R}^2}$ admits a reflection in its boundary if and only if it is a Jordan domain.

Main result

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Theorem
A Jordan domain D with $\infty \in \partial D$ is a John disk if and only if ∂D admits a reflection f such that

$$|f(z) - \zeta| \leq L\,|z - \zeta|$$

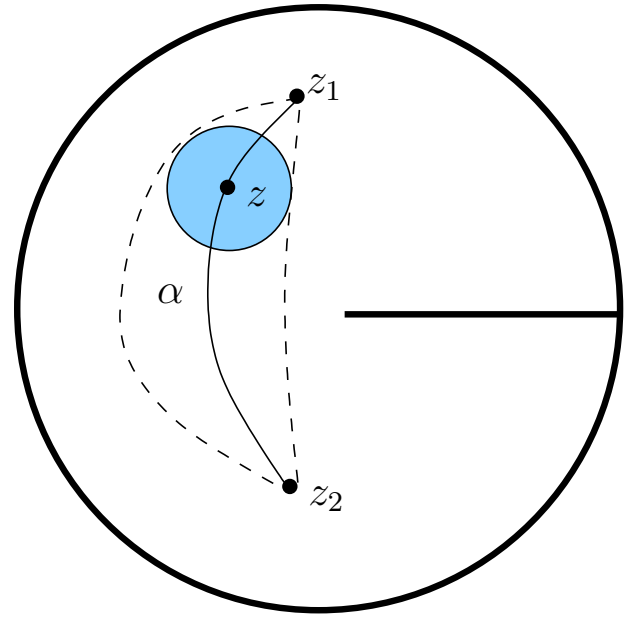
for all $z \in D, \zeta \in \partial D$.

John disks

Let $D \subset \mathbb{R}^2$ simply connected, proper subdomain.

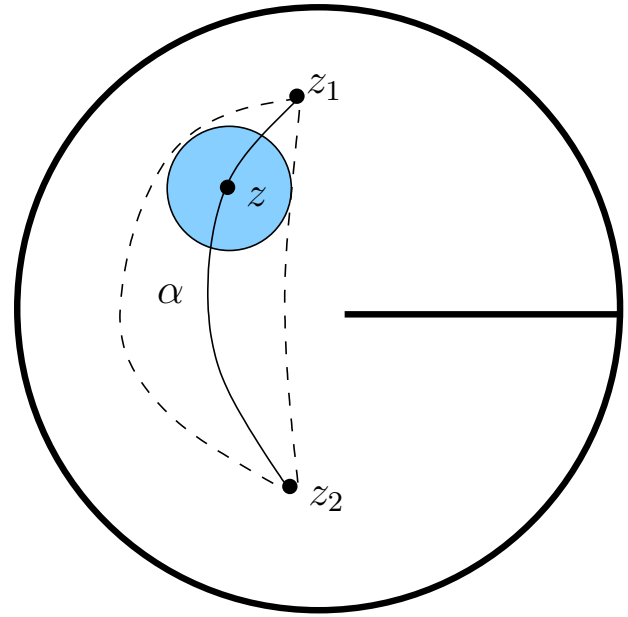
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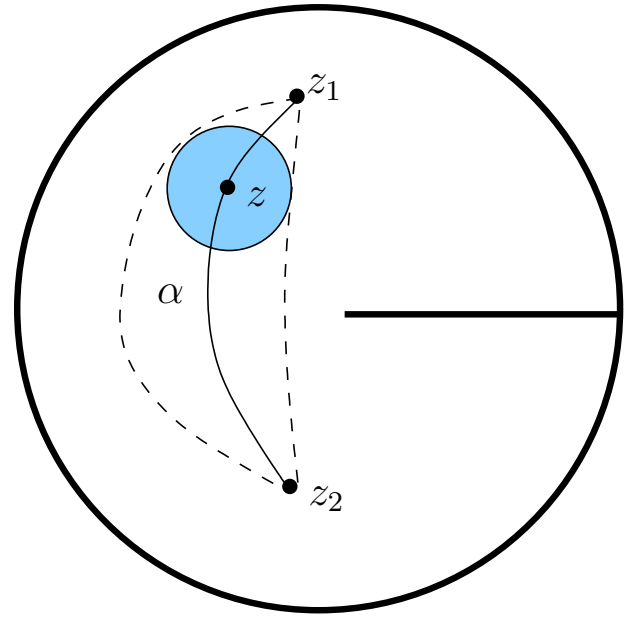
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D is a John disk if: $\exists\, c \geq 1$ such that all $z_1, z_2 \in D$ may be joined by an arc $\alpha \subset D$ such that for every $z \in \alpha$

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c\, d(z, \partial D)$$

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- describe auxiliary results about hyperbolic geodesics in John domains

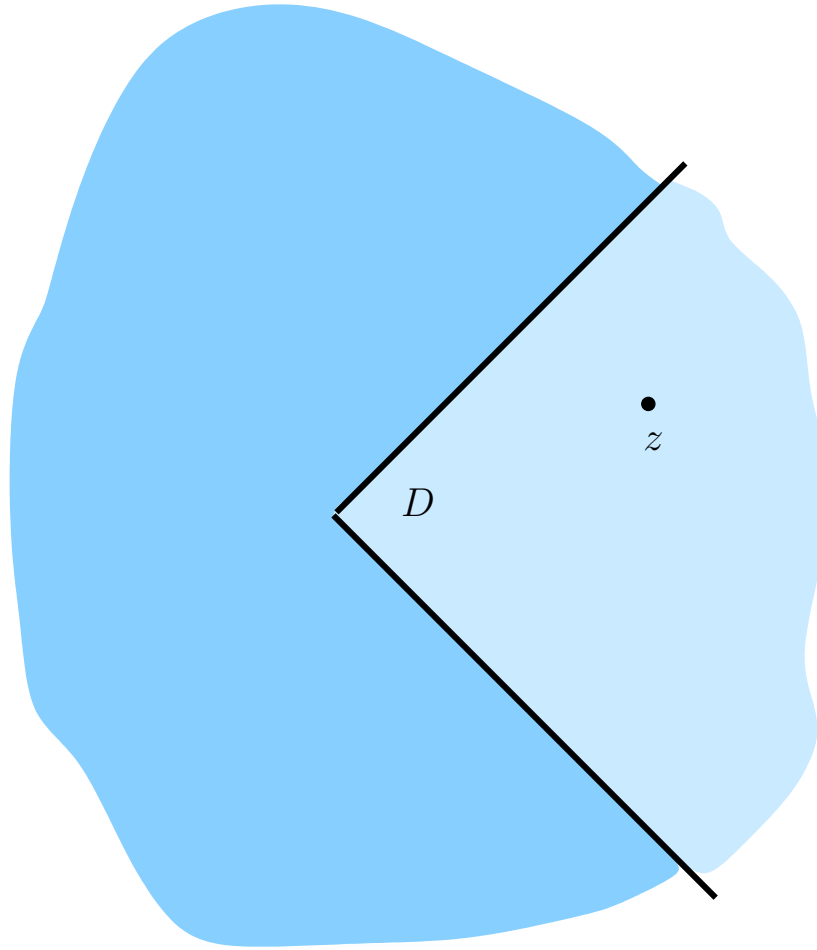
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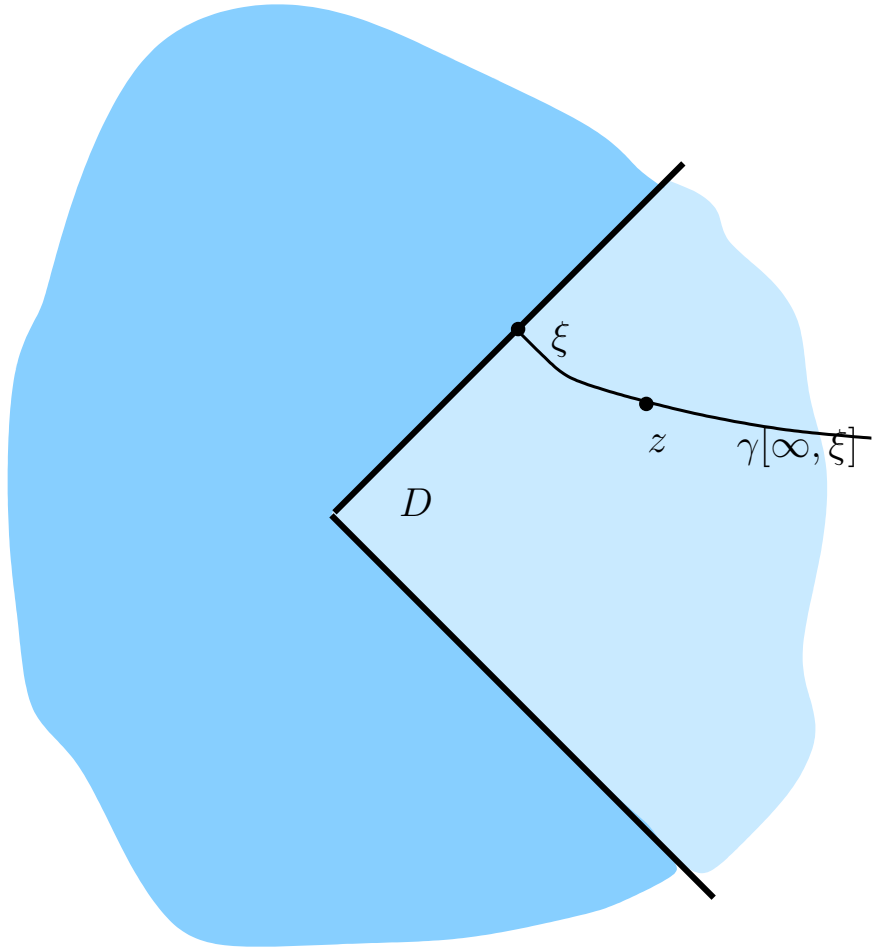
Assumption: D is Jordan domain in $\overline{\mathbb{R}^2}$

Reflections: construction



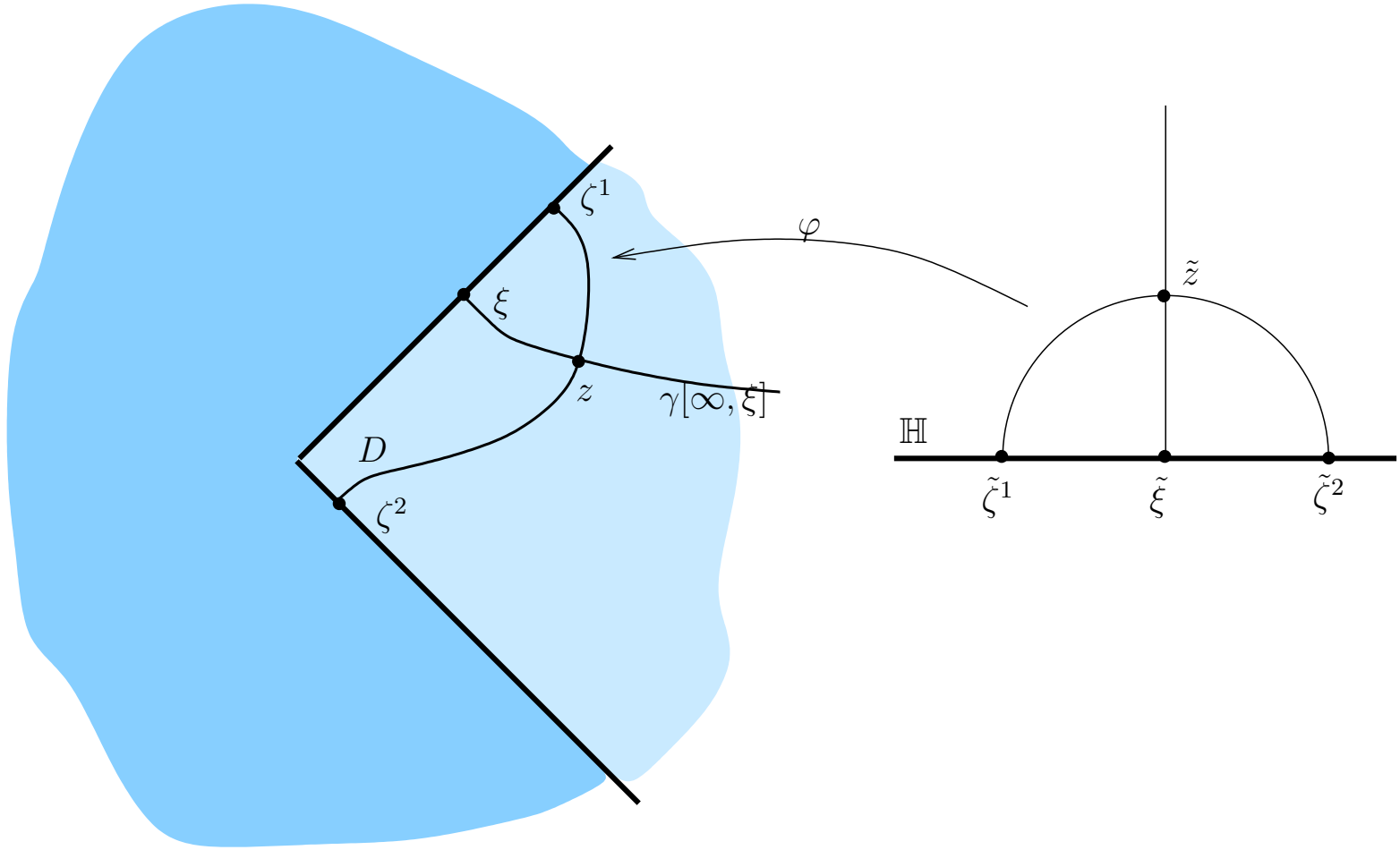


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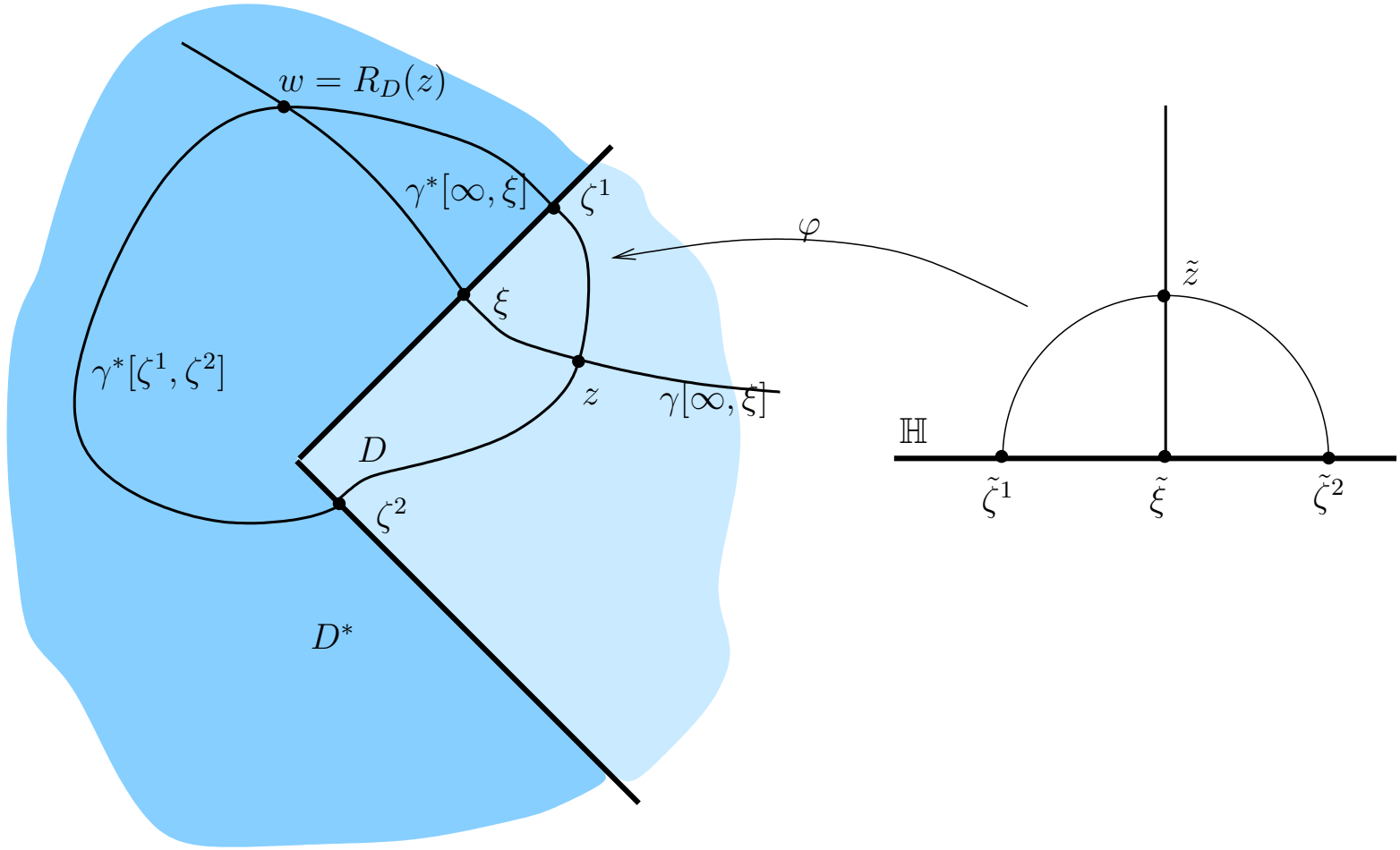




Reflections: construction



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$$w = R_D(z) = \begin{cases} \gamma^*[\infty, \xi] \cap \gamma^*[\zeta^1, \zeta^2], & \text{for } z \in D \\ z, & \text{for } z \in \partial D \\ \infty, & \text{for } z = \infty. \end{cases}$$

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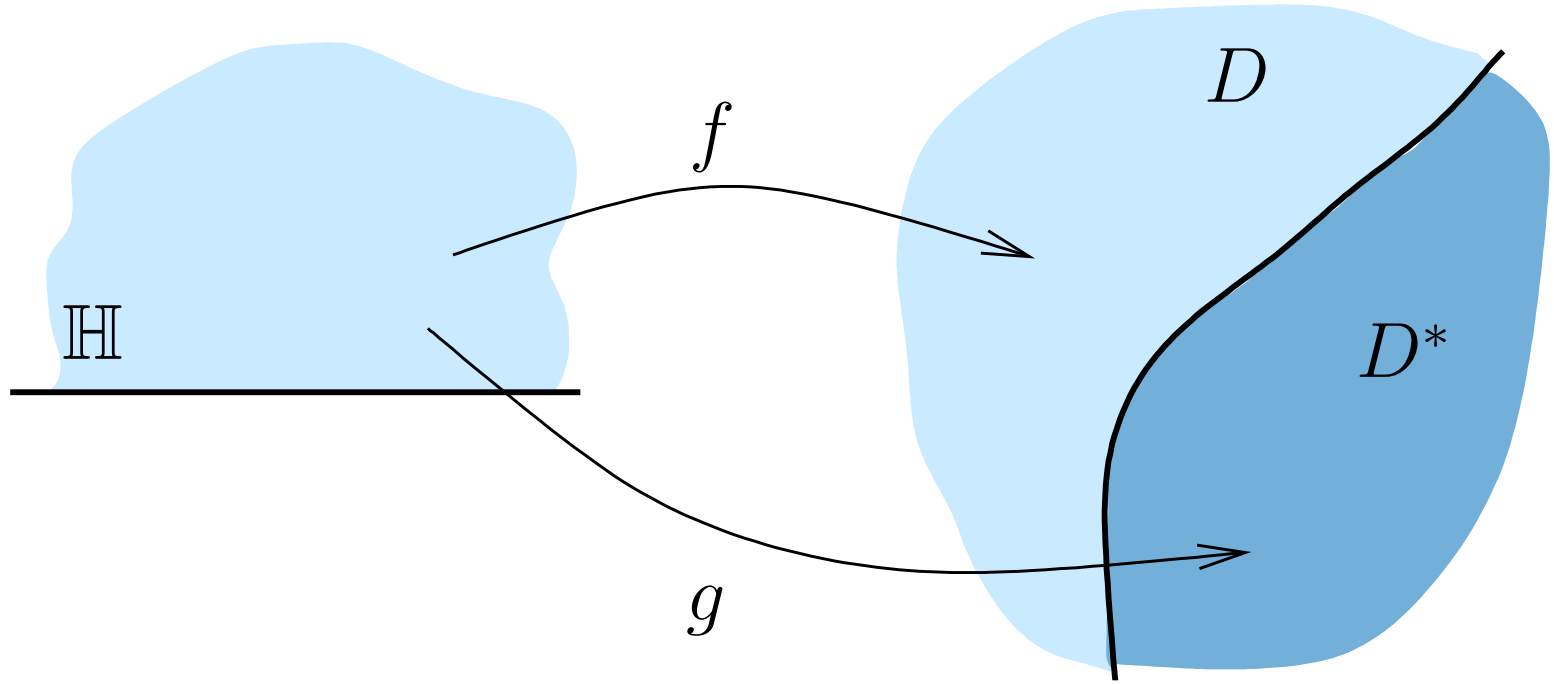
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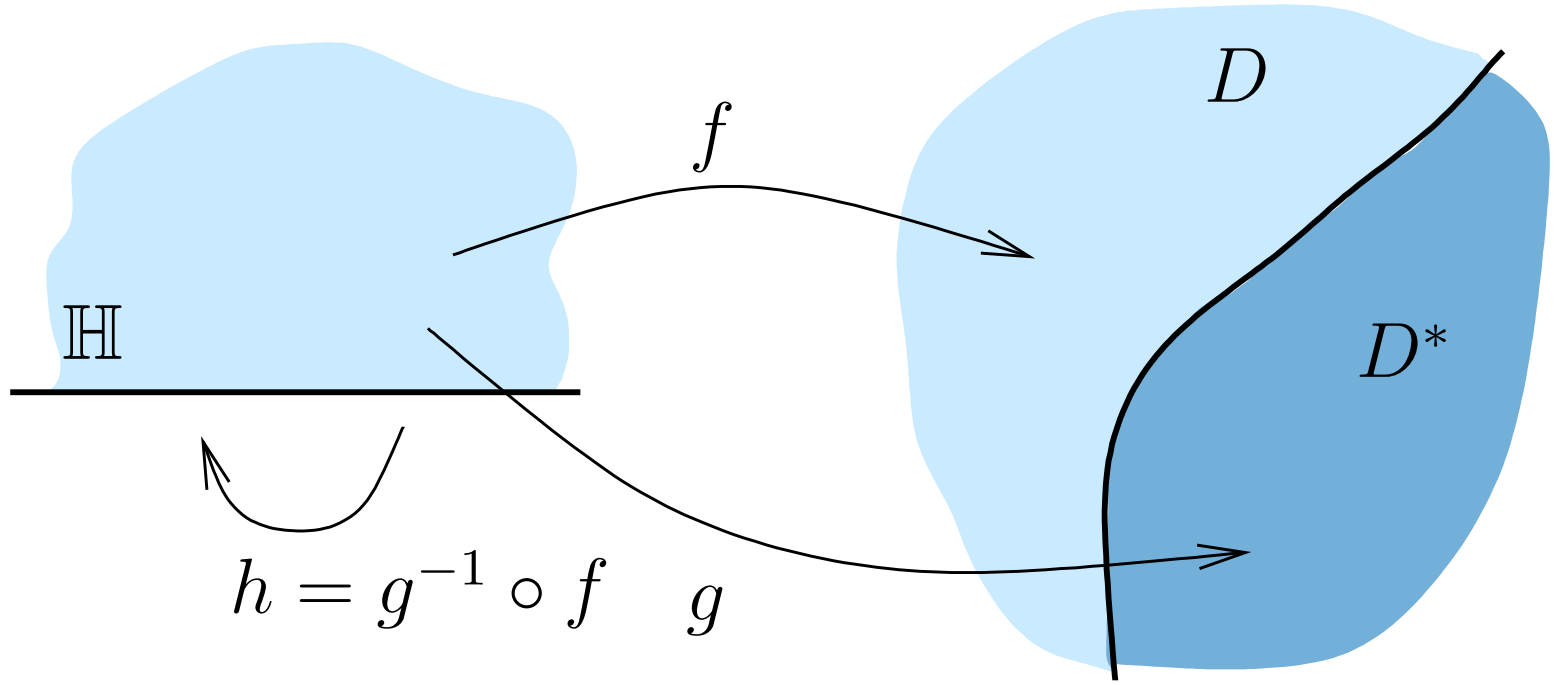
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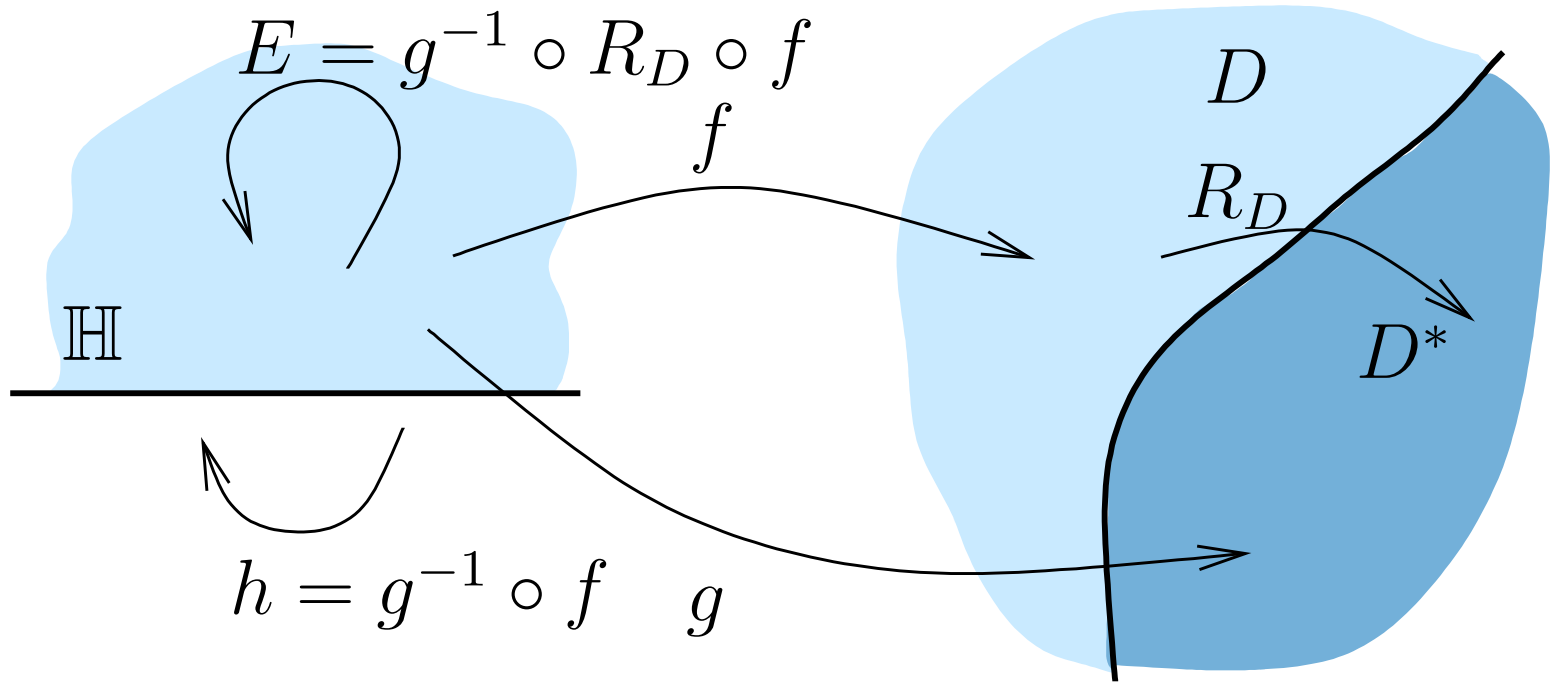
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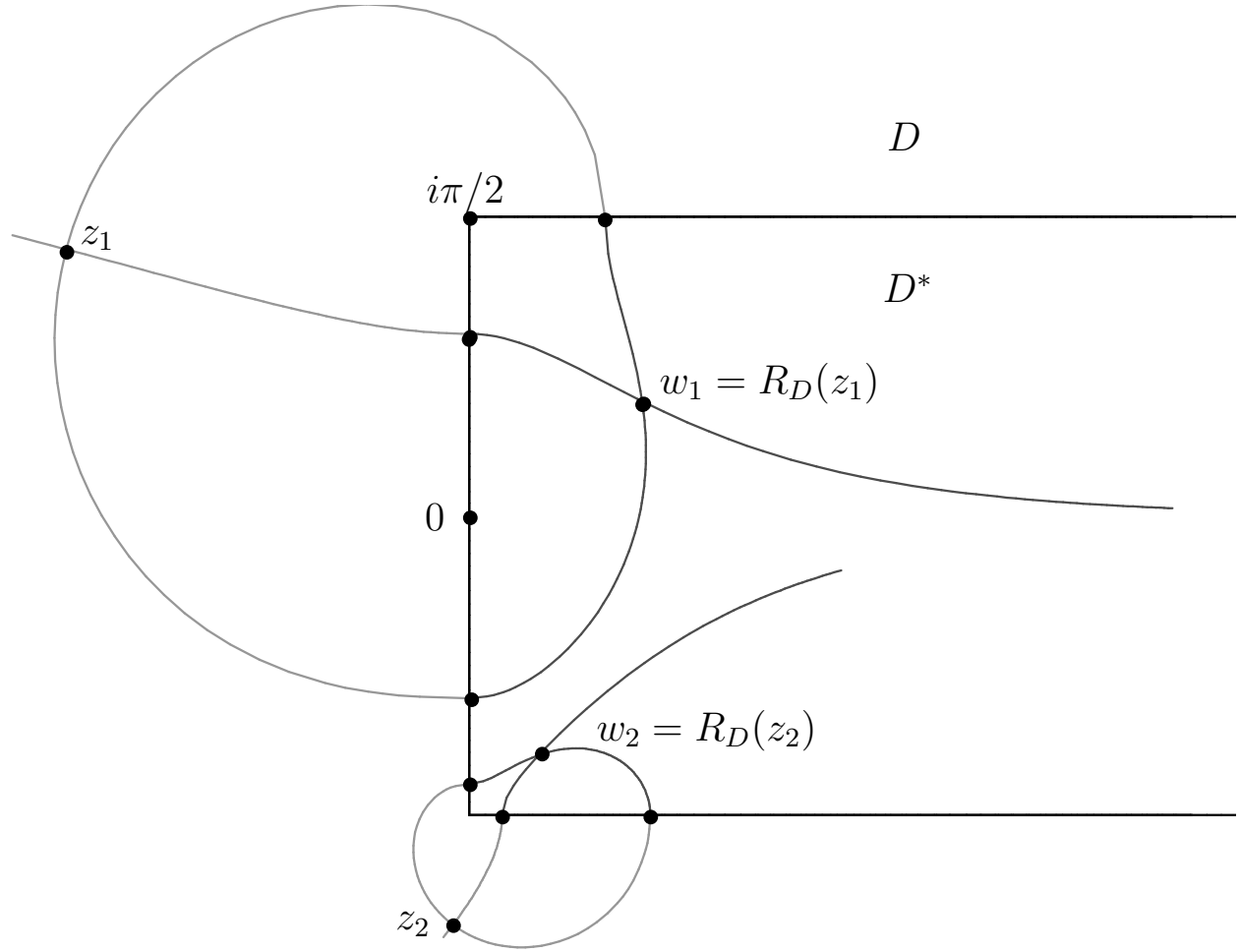
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Main result

A concrete example:



Hyperbolic geodesics in John disks

Theorem [Gehring-Hag 1987] *Simply connected proper subdomain* $D \subset \mathbb{R}^2$ *is* K*-quasidisk if and only if* $\exists \;\; c \geq 1$ *such that*

$$\frac{1}{c} \min_{j=1,2} |z_j - w| \leq |z - w| \leq c \max_{j=1,2} |z_j - w|$$

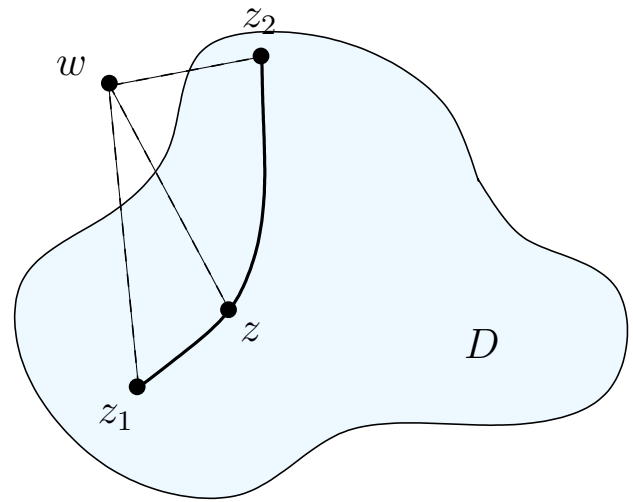
$\forall z \in \gamma[z_1, z_2], \forall w \notin D.$

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for every $w \notin D$ and every $z \in \gamma[z_1, z_2]$.

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Theorem *Jordan domain D is a John disk if and only if there is a constant b such that for $z_1, z_2 \in \overline{D}$*

$$\min_{j=1,2} \lambda_D(z_j, w) \leq b\lambda_D(z, w),$$

for every $w \in \partial D$, every $z \in \gamma[z_1, z_2]$.

Internal distances

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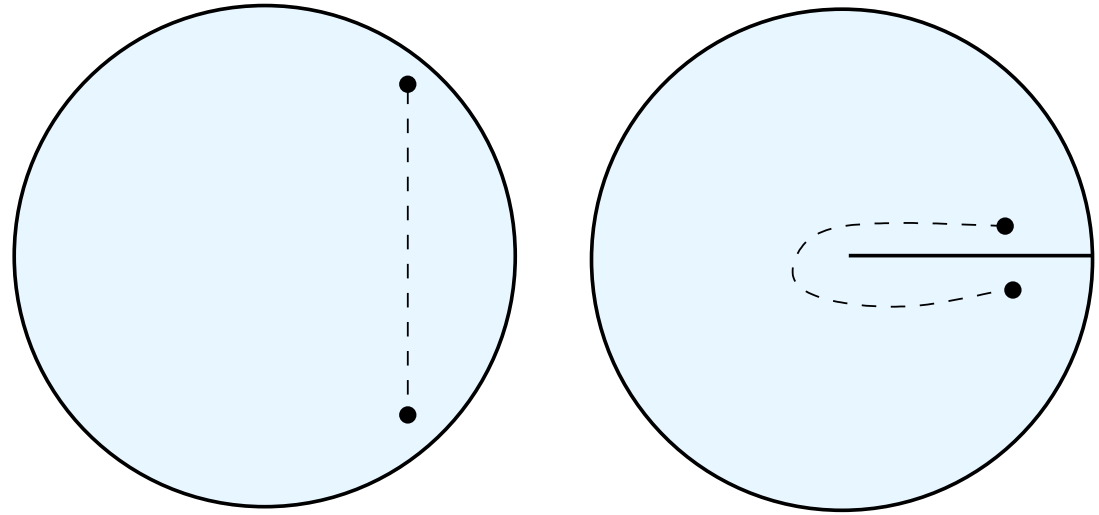
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Theorem *Let $D \subset \mathbb{R}^2$ Jordan domain. D^* is a John disk if and only if there is a constant b such that for any two $z_1, z_2 \in \overline{D}$ and any $w \notin D$,*

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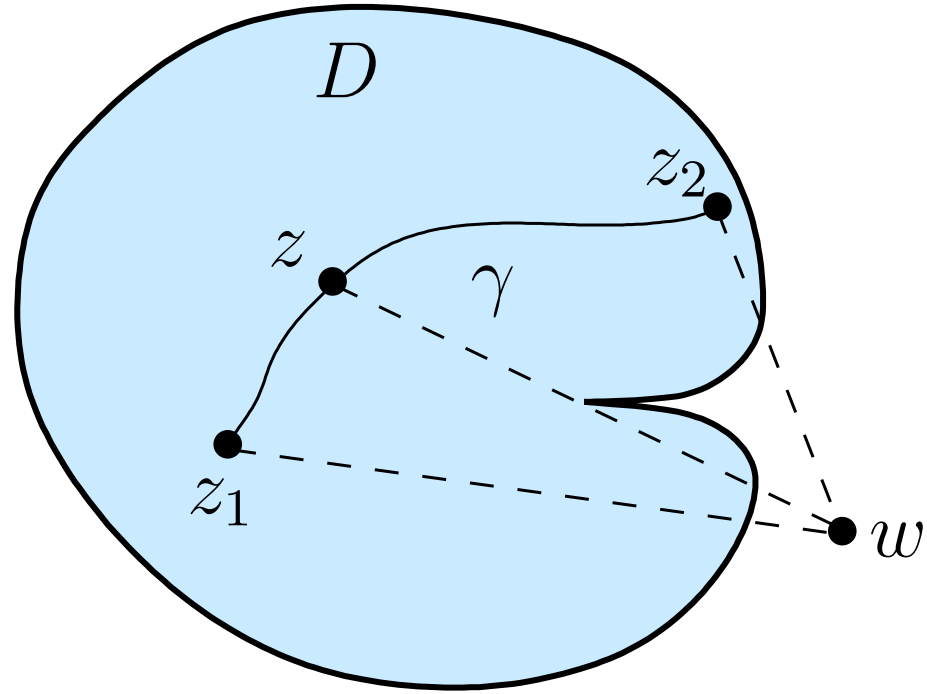
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Use construction, min-max properties and quasisymmetry:

$$f : \mathbb{H} \to (D, \lambda_D)$$

Proof of main result (sufficiency)

Suppose that ∂D admits a reflection $f : D \to D^*$ with

$$|f(z) - \zeta| \leq L\,|z - \zeta|, \quad \forall z \in D, \zeta \in \partial D.$$

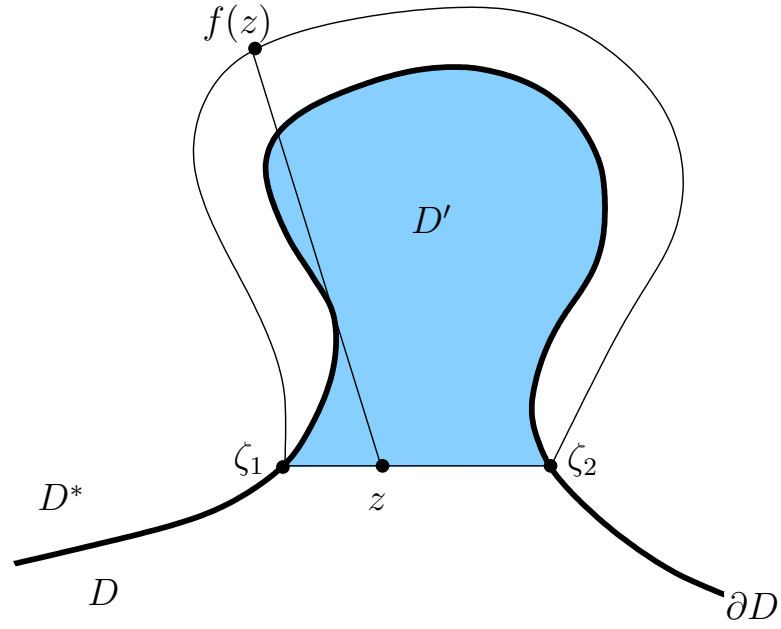
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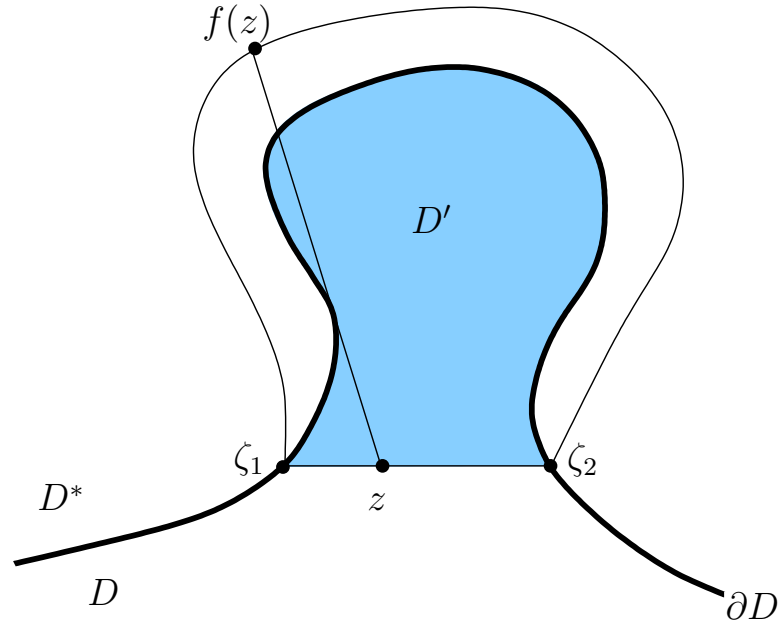
Remember: $|f(z) - \zeta_i| \leq L|z - \zeta_i|$

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Thus

$$\operatorname{diam} D' \leq 2\,L\,|\zeta_1 - \zeta_2|$$

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- Is R_D quasiconformal if D is a quasidisk?
- Can R_D be used to study other domains as well?