Geometry of John disks

Ole Jacob Broch

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- John disks are one sided quasidisks

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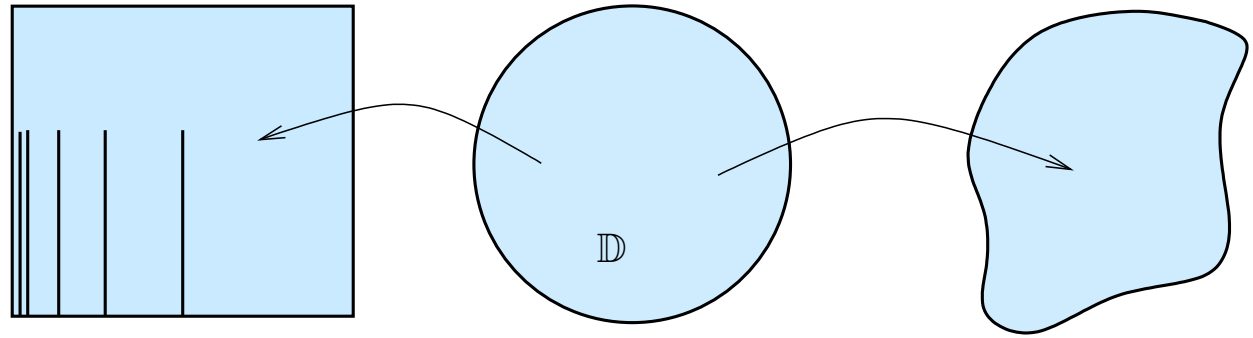
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All are related to the geometry - “the shape” - of D.
All have “classical” roots.
Will provide some background and motivation first.

Intro: Riemann mapping theorem

For any simply connected domain D there is a *conformal* map of the unit disk onto D:

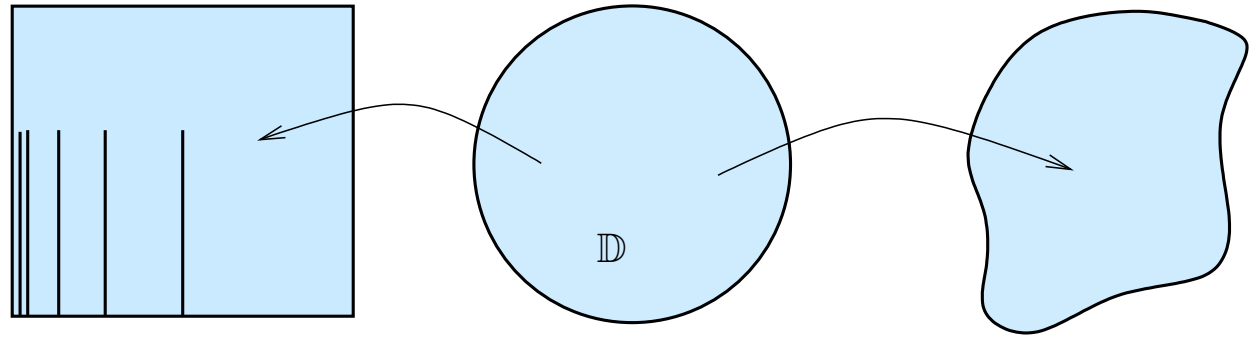
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One relevant question can be: what is the connection between the "shape" of D and the properties of f?

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Examples:

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- hyperbolic (non-euclidean) distance:

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- modulus of a curve family or a quadrilateral

$$\operatorname{mod} f(\Gamma) = \operatorname{mod} \Gamma.$$

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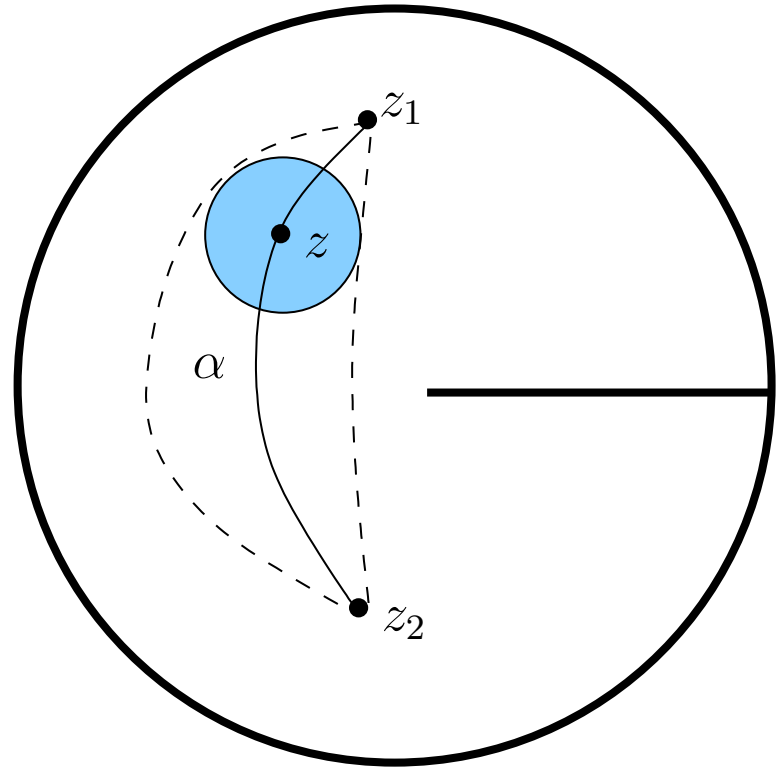
- What (if any) are the relations between euclidean quantities and conformal invariants in D?
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- Are there any properties of D that makes it possible to "do analysis" in it?
  - For example: what can be said about certain types of functions in D?

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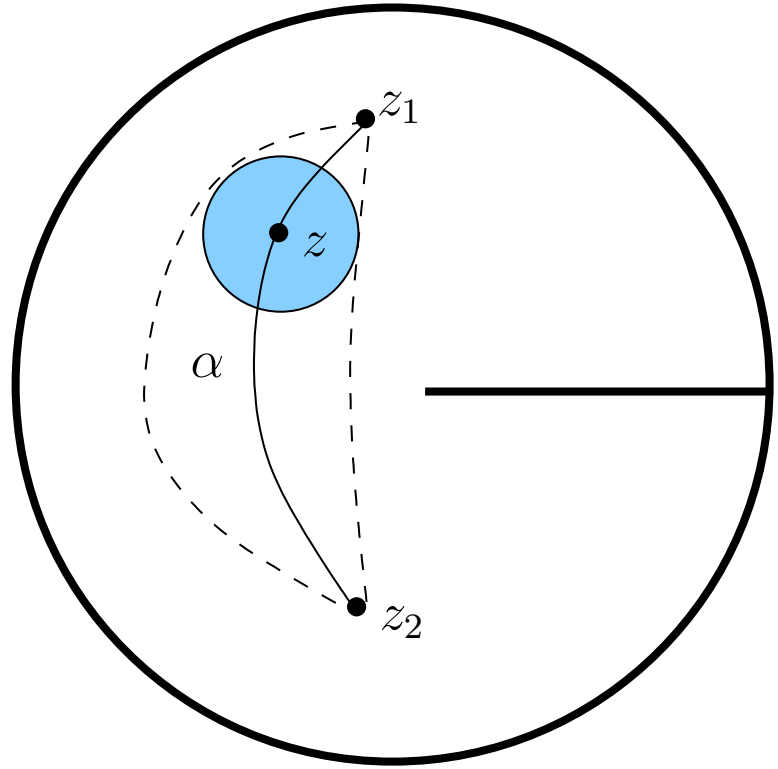
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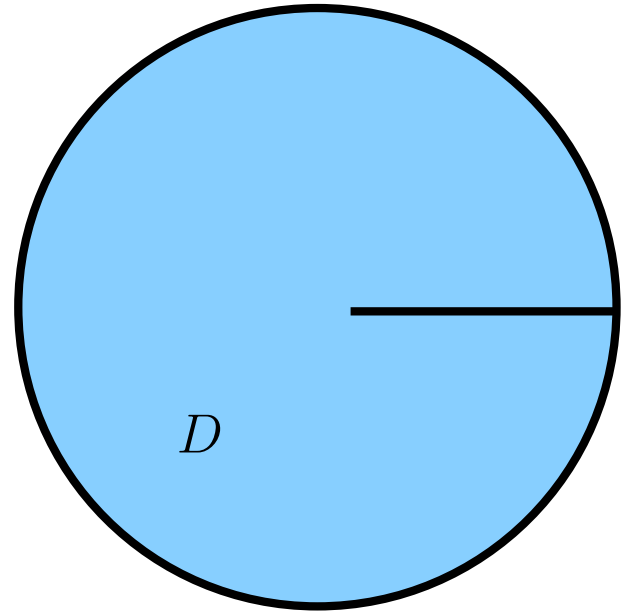
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D is a John disk if: $\exists\, c \geq 1$ such that all $z_1, z_2 \in D$ may be joined by an arc $\alpha \subset D$ such that

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c\, d(z, \partial D)$$

Introduction: Examples

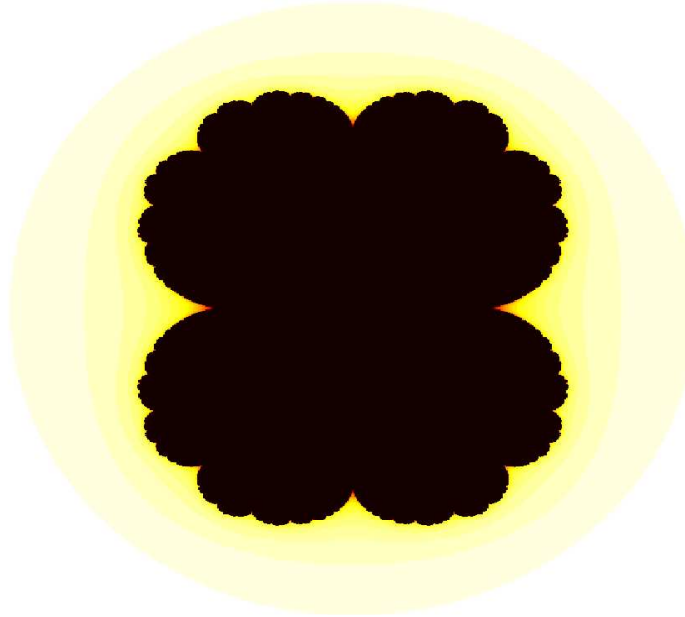


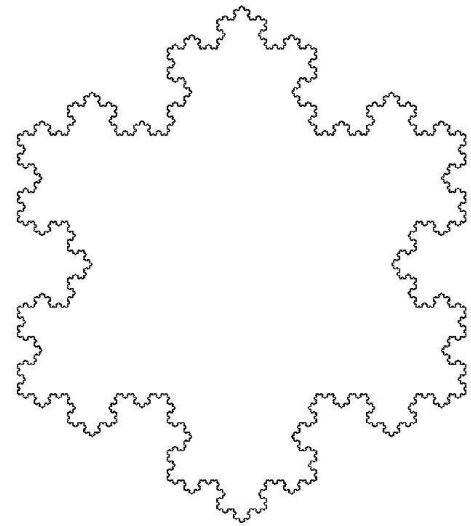
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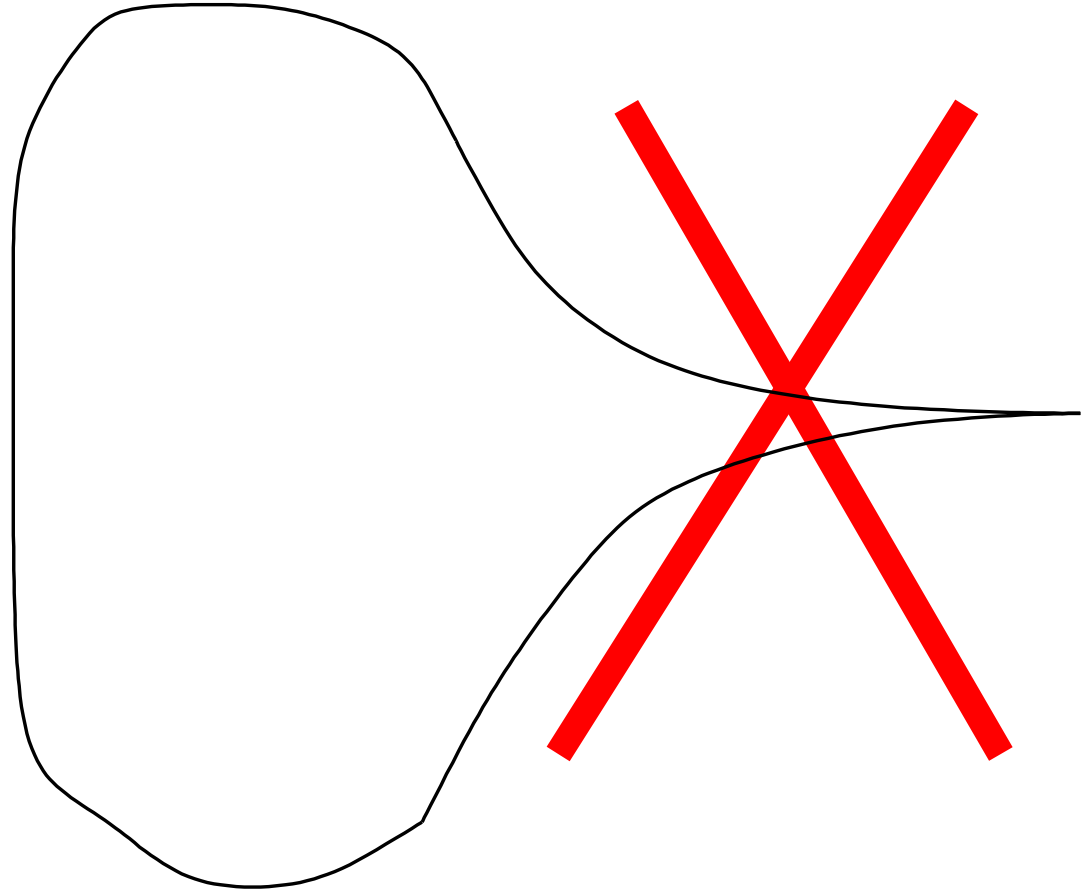
Introduction: Examples

Many domains with fractal boundaries are John disks:





Introduction: *Not* a John domain



Outward cusps are not allowed!

Introduction: Internal distances

D domain. Internal distance between $z_1, z_2 \in D$:

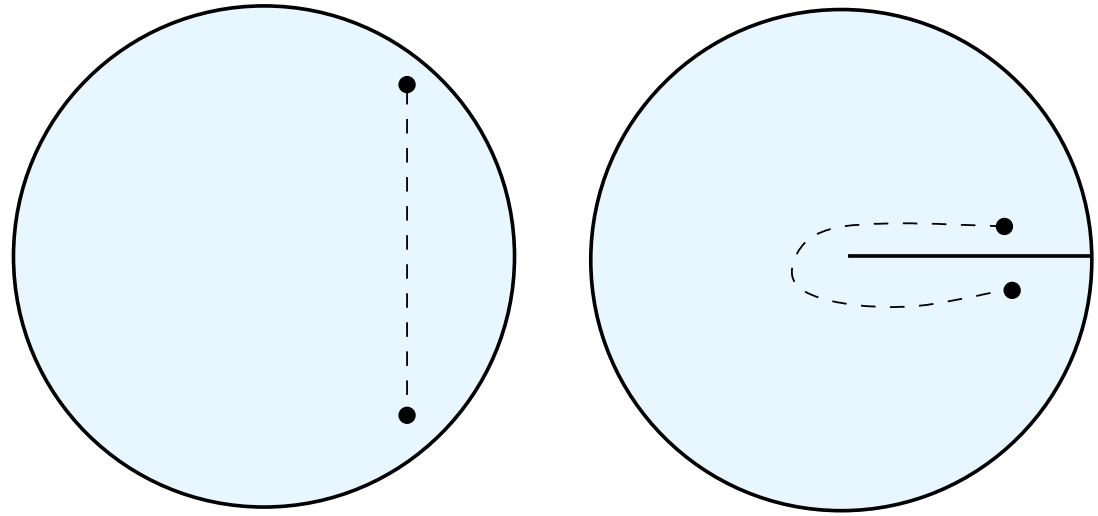
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If we use internal distances in domains with inward cusps, the domain "looks like a disk from the inside":

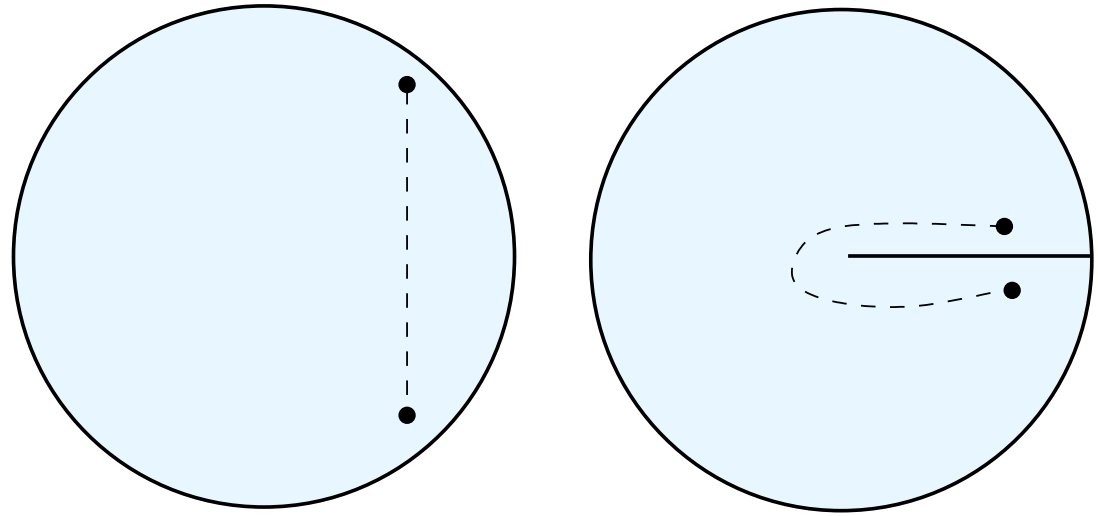


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Will use the metric spaces:

$$(D, \lambda_D), \ (\partial D, \lambda_D), \ (\overline{D}, \lambda_D)$$

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A quasidisk is the image of a disk or a half plane under a quasiconformal map

$$f : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}.$$

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A QC map $f : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ (with $f(\infty) = \infty$) satisfies

$$\frac{|f(z_1) - f(z_2)|}{|f(z_2) - f(z_3)|} \leq C \text{ when } \frac{|z_1 - z_2|}{|z_2 - z_3|} \leq 1$$

for $z_1, z_2, z_3 \in \mathbb{R}^2$, $z_2 \neq z_3$.

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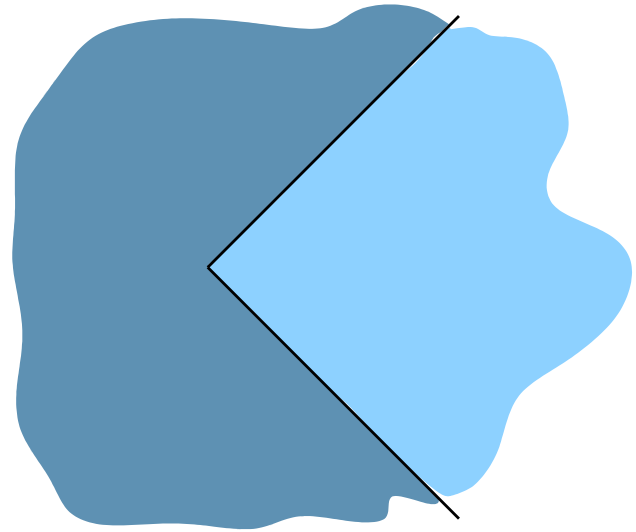
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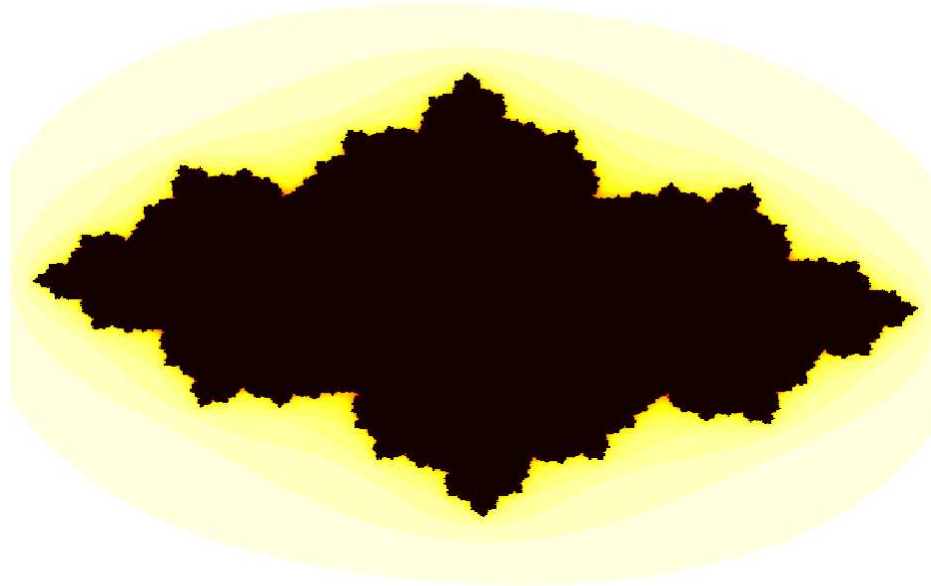
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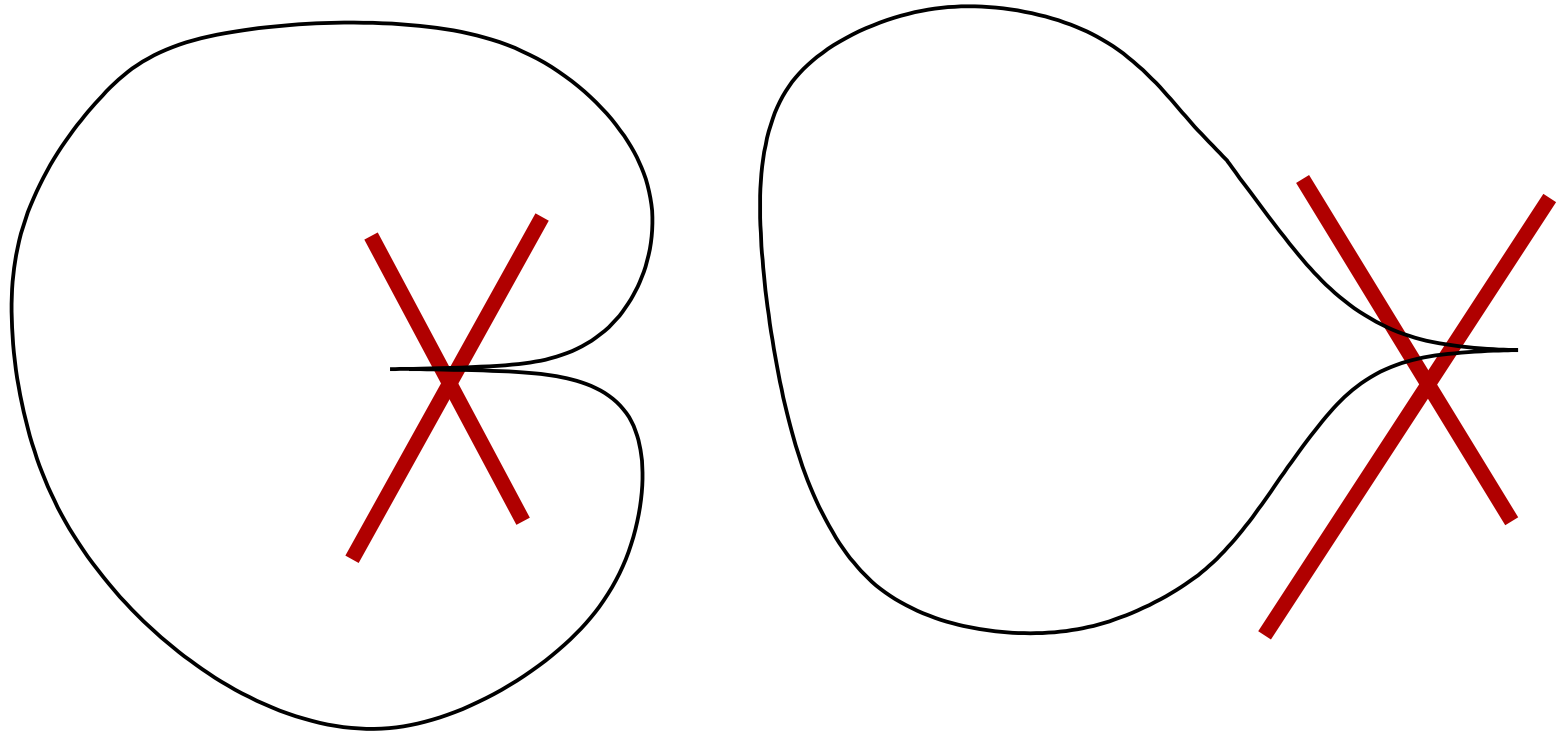
Thus quasiconformal maps are quasisymmetric

Introduction: Quasidisks





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Thus: have to describe John disks intrinsically (in it-self, without thinking about outside)

Intro: The three point property

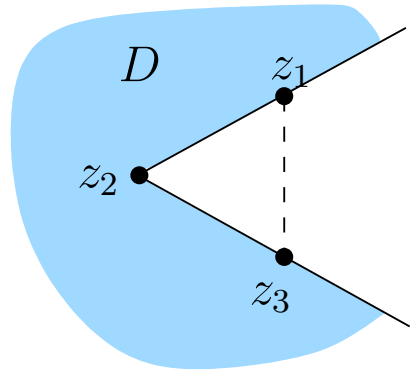
Quasidisk:

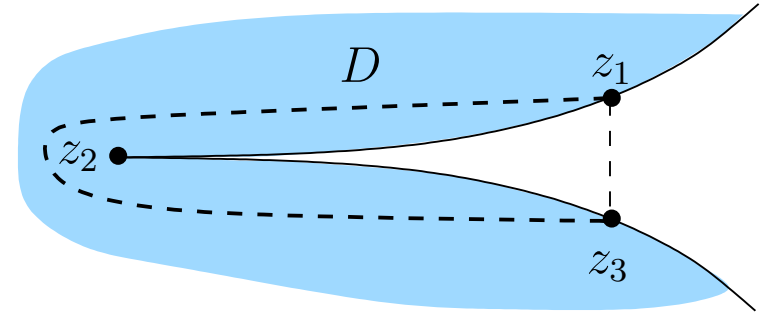
$$|z_1 - z_2| \leq c|z_1 - z_3|$$

John disk:

$$\lambda_D(z_1, z_2) \leq c\lambda_D(z_1, z_3).$$

In both cases: $z_1, z_3 \in \partial D$. z_2 in that component of $\partial D \setminus \{z_1, z_3\}$ of least Euclidean diameter.





Extension of bilipschitz maps

The Schoenflies theorem says:
Any homeomorphism

$$f : \mathbb{T} \to J$$

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What if we impose conditions on f? For example, what if:

- f is quasisymmetric
- f is bilipschitz

Extension of bilipschitz maps

A homeomorphism $f : A \to B$ is L-bilipschitz if

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Theorem (Gehring 1982/86) Let D be a (bounded) Jordan domain. Then D is a quasidisk if and only if every bilipschitz map $f : \partial D \to \mathbb{R}^2$ has a bilipschitz extension $\tilde{f} : \overline{D} \to \mathbb{R}^2$.

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is L-bilipschitz, then D' is a c'-John domain, and f extends as an M-bilipschitz map

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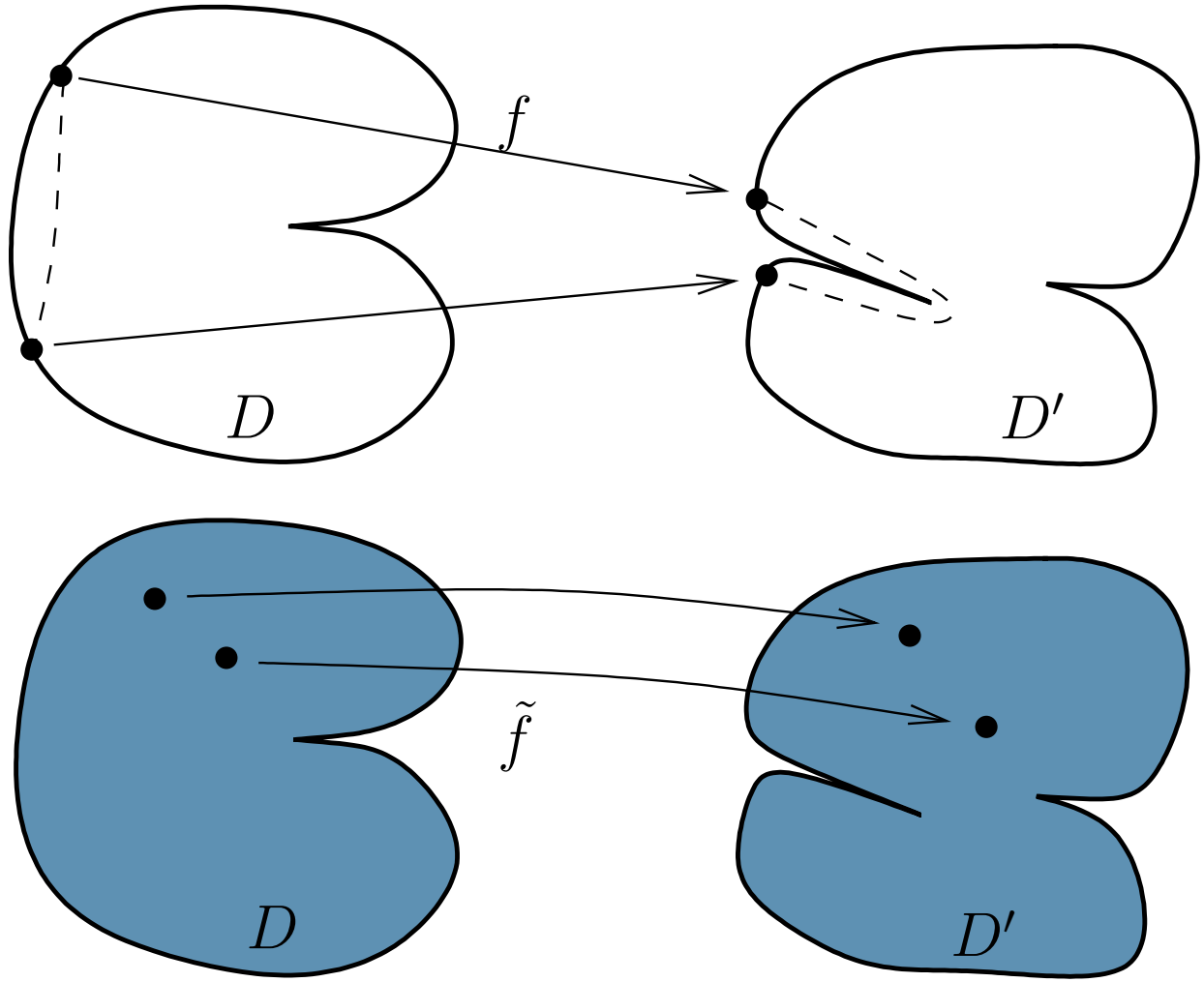
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$$\tilde{f} : (\overline{D}, \lambda_D) \to (\overline{D'}, \lambda_{D'}).$$

Remember: $f : (\partial D, \lambda_D) \to (\partial D', \lambda_{D'})$ is bilipschitz if

$$L^{-1} \lambda_D(z_1, z_2) \leq \lambda_{D'}(f(z_1), f(z_2)) \leq L \lambda_D(z_1, z_2).$$

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- controlled distortion of distance to the boundary

□

Quadrilaterals and John disks

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Quadrilaterals and John disks

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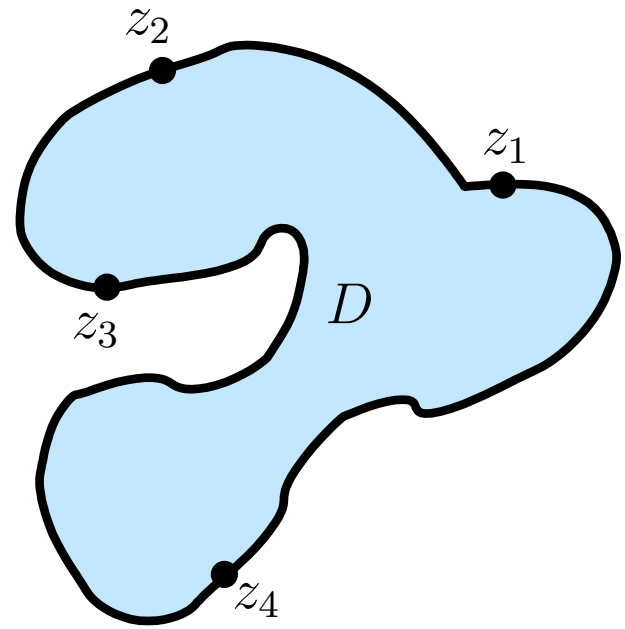
Quadrilaterals and John disks

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Notation:

$$Q = D(z_1, z_2, z_3, z_4).$$

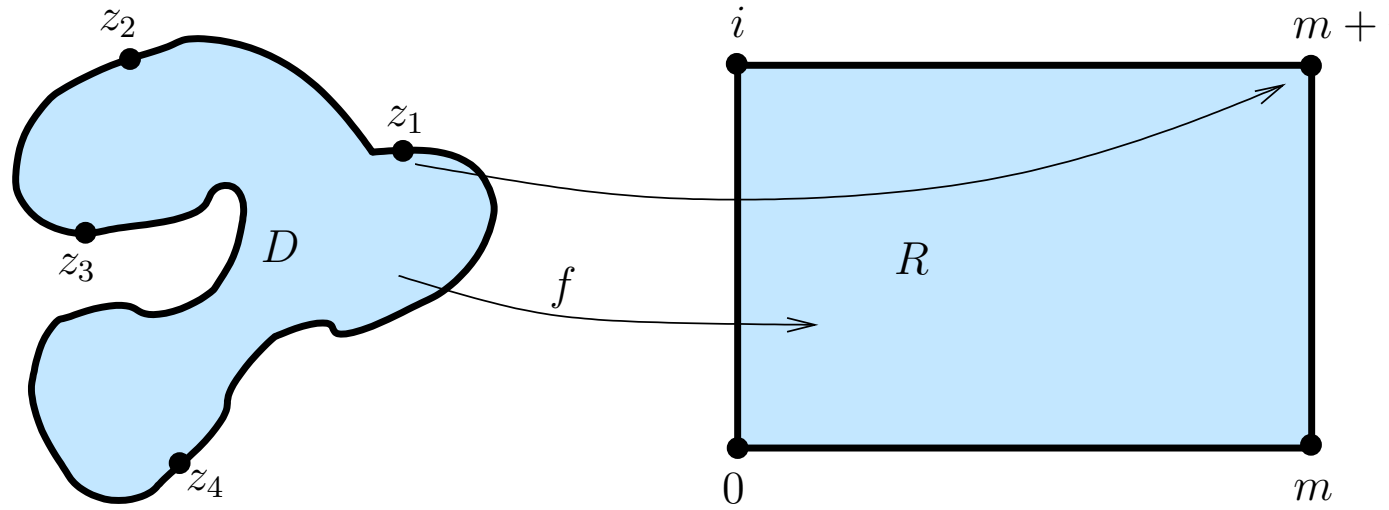


Quadrilaterals and John disks

The modulus of a quadrilateral $Q = D(z_1, z_2, z_3, z_4)$ is the unique number m such that there is a conformal map

$$f : D \to R = \{z = x + iy : 0 < x < m, 0 < y < 1,\}$$

with $f(z_1) = m + i$, $f(z_2) = i$, $f(z_3) = 0$ and $f(z_4) = m$.



Quadrilaterals and John disks

If $D = \mathbb{D}$, then we can compute the modulus of $\mathbb{D}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$:

$$\operatorname{mod} \mathbb{D}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \frac{2}{\pi} \mu(|\zeta_1, \zeta_2, \zeta_3, \zeta_4|^{-1/2}).$$

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Here the modulus is increasing with $|\zeta_1, \zeta_2, \zeta_3, \zeta_4|$;

$$|\zeta_1, \zeta_2, \zeta_3, \zeta_4| = \frac{|\zeta_1 - \zeta_3|}{|\zeta_1 - \zeta_4|} \frac{|\zeta_2 - \zeta_4|}{|\zeta_2 - \zeta_3|}.$$

Quadrilaterals and John disks

If $D = \mathbb{D}$, then we can compute the modulus of $\mathbb{D}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$:

$$\operatorname{mod} \mathbb{D}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \frac{2}{\pi} \mu(|\zeta_1, \zeta_2, \zeta_3, \zeta_4|^{-1/2}).$$

Here the modulus is increasing with $|\zeta_1, \zeta_2, \zeta_3, \zeta_4|$;

$$|\zeta_1, \zeta_2, \zeta_3, \zeta_4| = \frac{|\zeta_1 - \zeta_3|}{|\zeta_1 - \zeta_4|} \frac{|\zeta_2 - \zeta_4|}{|\zeta_2 - \zeta_3|}.$$

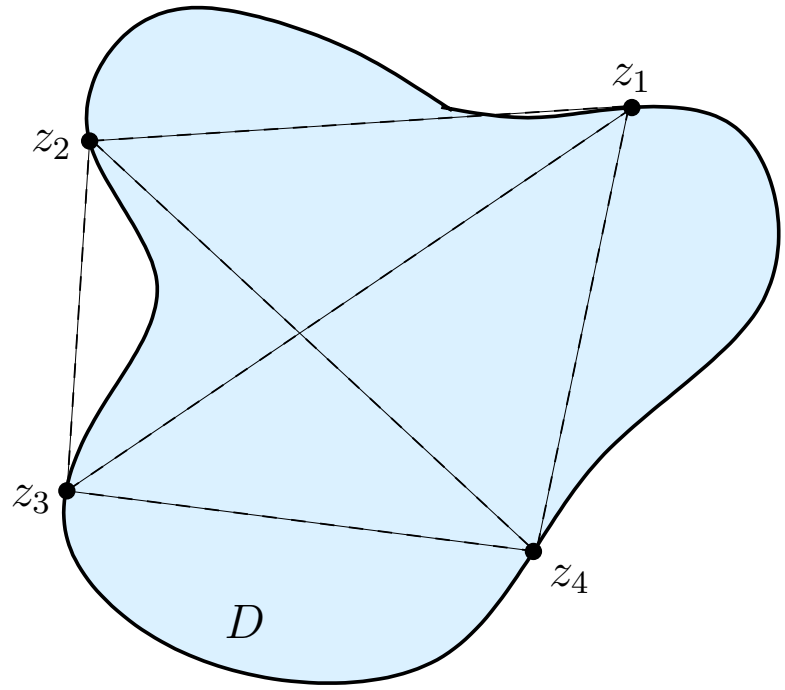
Question: Can we compute the modulus is a similar way in other domains? Can we at least find a bound for the modulus in terms the cross ratio?

Quadrilaterals and John disks

Theorem 3.4.3 A Jordan domain D is a quasidisk if and only if there is some increasing function $\nu\colon [0,\infty) \to [0,\infty)$ with

$$\operatorname{mod} D(z_1, z_2, z_3, z_4) \le \nu(|z_1, z_2, z_3, z_4|)$$

for every ordered quadruple $z_1, z_2, z_3, z_4 \in \partial D$.

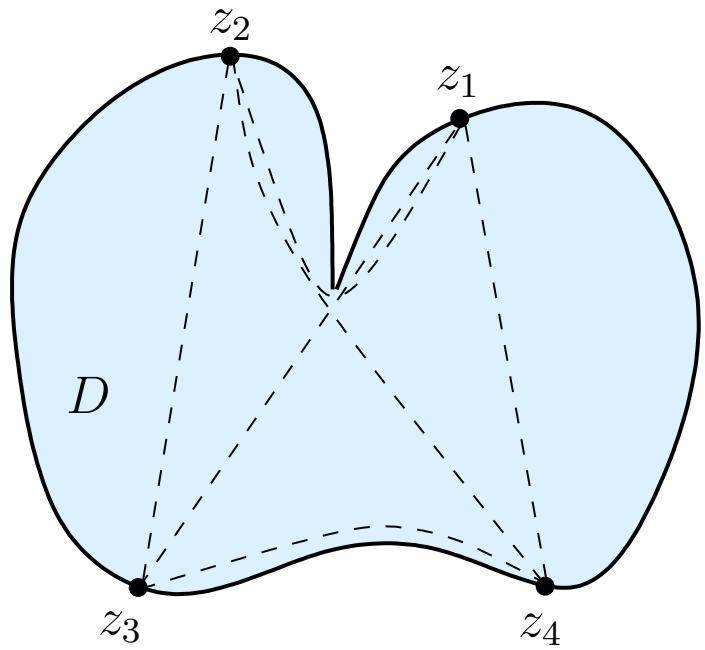


Quadrilaterals and John disks

Theorem 3.4.3 A Jordan domain D is a John disk if and only if there is an increasing function $\nu\colon [0,\infty) \to [0,\infty)$ with

$$\operatorname{mod} D(z_1, z_2, z_3, z_4) \le \nu(|z_1, z_2, z_3, z_4|_\lambda)$$

for every ordered quadruple $z_1, z_2, z_3, z_4 \in \partial D$.



$$|z_1, z_2, z_3, z_4|_\lambda = \frac{\lambda_D(z_1, z_3)}{\lambda_D(z_1, z_4)} \frac{\lambda_D(z_2, z_4)}{\lambda_D(z_2, z_3)}$$

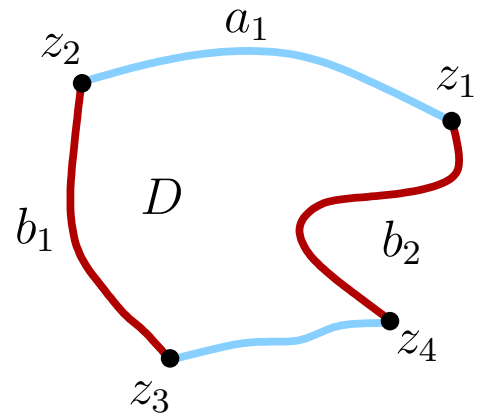
Quadrilaterals and John disks

Theorem 3.5.13 A bounded Jordan domain D is a John disk if and only if there is a constant c such that whenever a_1, b_1, a_2, b_2 are consecutive boundary arcs with

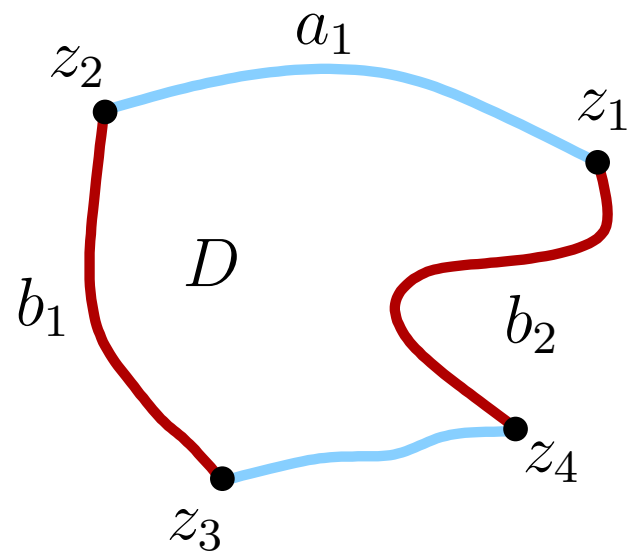
$$\operatorname{diam}(a_1) \le 2 \min_{j=1,2} \operatorname{diam}(b_j)$$

we have

$$\operatorname{mod} D(z_1, z_2, z_3, z_4) \le c.$$



Quadrilaterals and John disks



Proof: I prove that the property in the Theorem is equivalent to the harmonic doubling condition, which is characteristic for John disks.

□

Reflections

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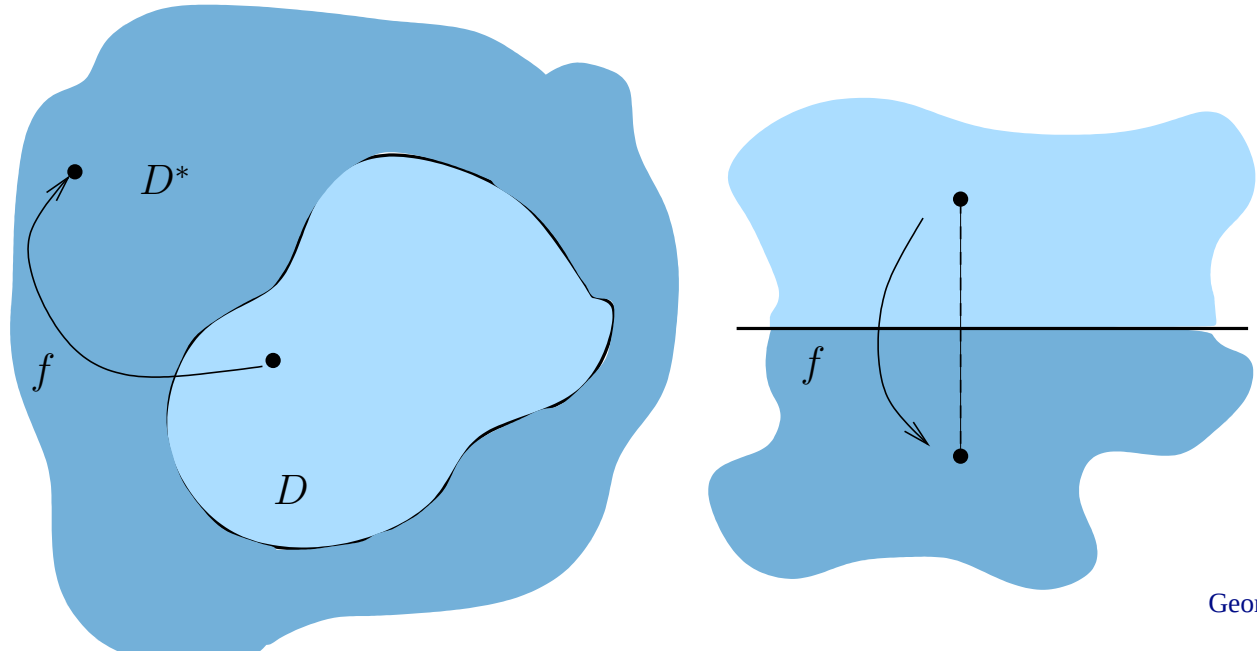
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Reflections: earlier results

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Ahlfors (1963): A Jordan domain D admits a quasiconformal reflection in its boundary if and only if it is a quasidisk.

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Reflections: earlier results

Ahlfors (1963): A Jordan domain D admits a quasiconformal reflection in its boundary if and only if it is a quasidisk.

Ahlfors (1966): A Jordan domain D with $\infty \in \partial D$ admits a bilipschitz reflection if and only if it is a quasidisk.

Gehring and Hag (2001): A domain $D \subset \overline{\mathbb{R}^2}$ admits a reflection in its boundary if and only if it is a Jordan domain.

Reflections: a new result

Is there some result describing John disks in terms of reflections?

Reflections: a new result

Is there some result describing John disks in terms of reflections?

Theorem 4.4.5 A Jordan domain D with $\infty \in \partial D$ is a John disk if and only if ∂D admits a reflection f such that

$$|f(z) - \zeta| \leq L|z - \zeta|$$

for all $z \in D, \zeta \in \partial D$.

Reflections

Suppose D is a John disk.

Difficulty: prove that ∂D admits reflection f with

$$|f(z) - \zeta| \leq L|z - \zeta|.$$

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- use hyperbolic geodesics to move from D to D^*
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Reflections

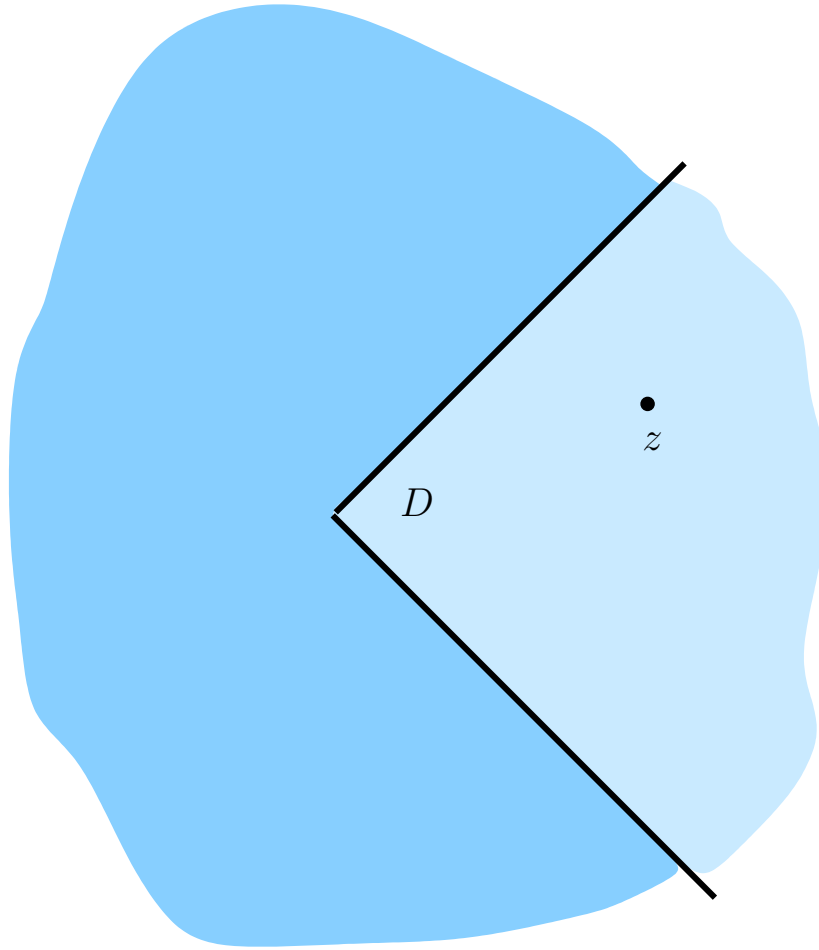
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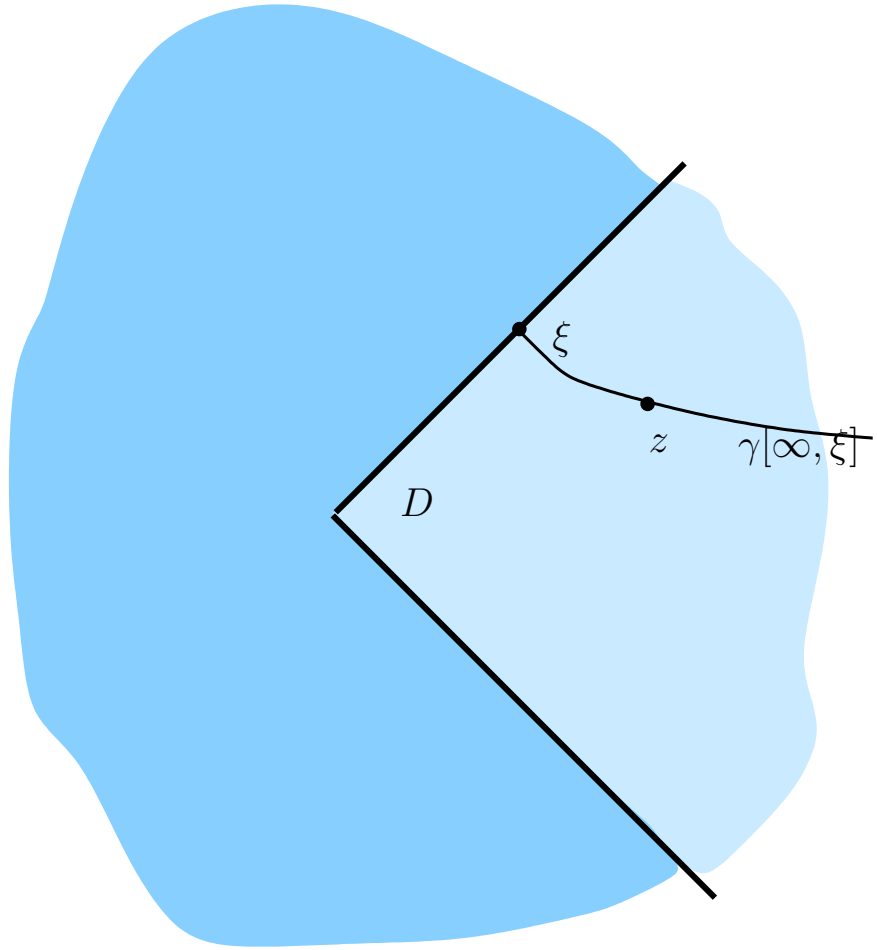
- problem solved by explicit construction
- use hyperbolic geodesics to move from D to D^*
- compare properties of hyperbolic geodesics in D and D^*
- need to prove auxiliary results about geodesics in D and D^*.

Reflections: construction



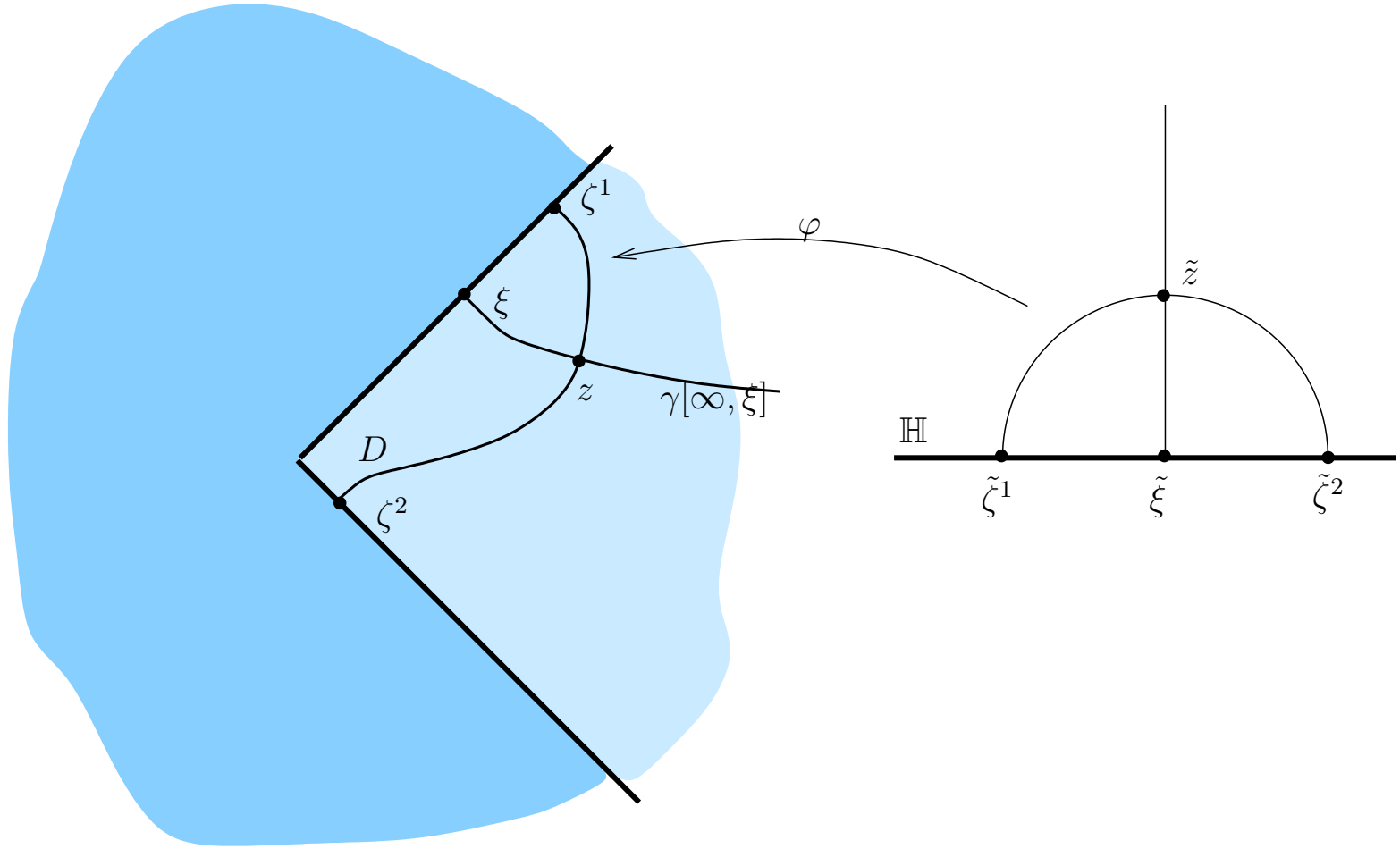


Reflections: construction

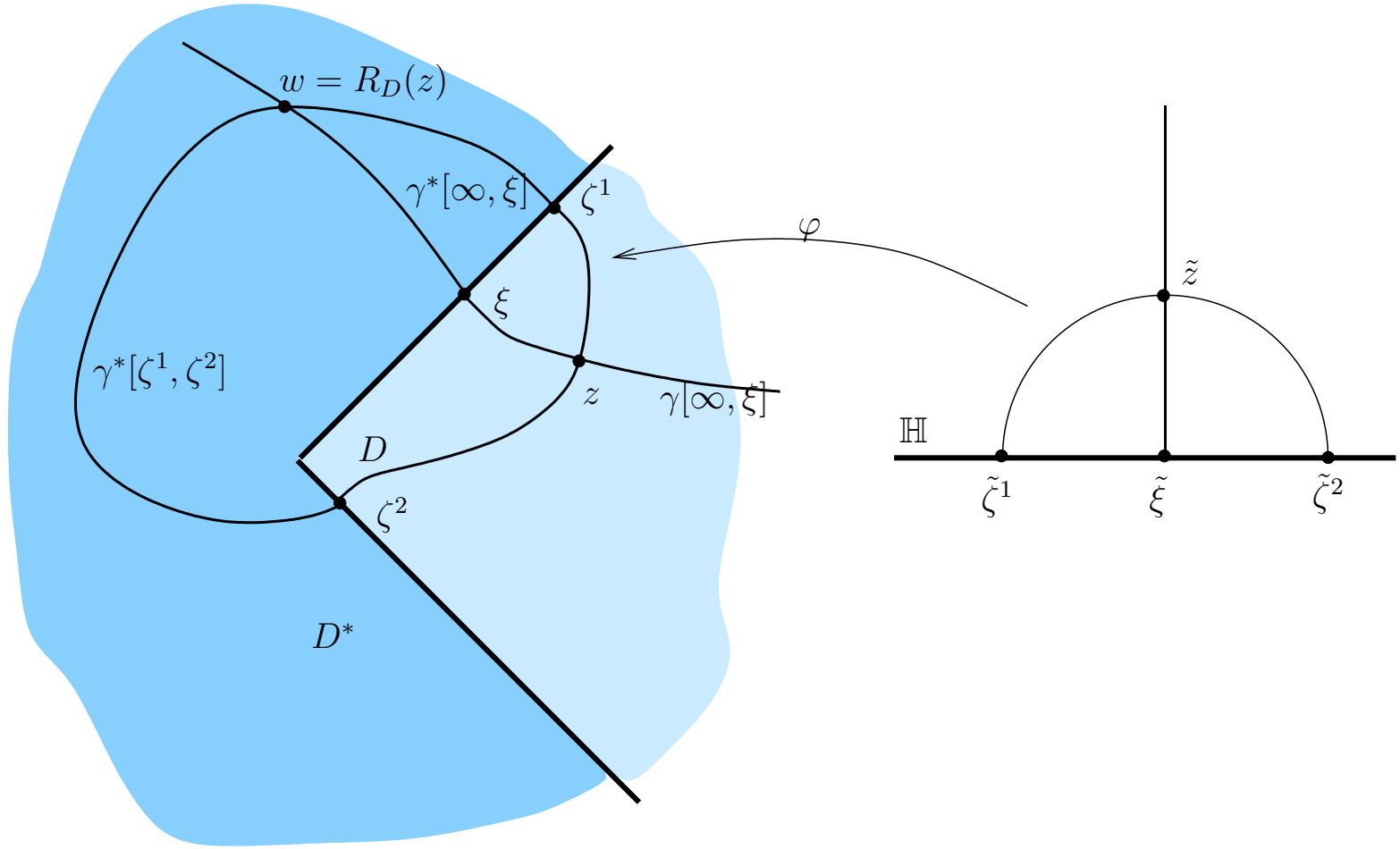




Reflections: construction



Reflections: construction



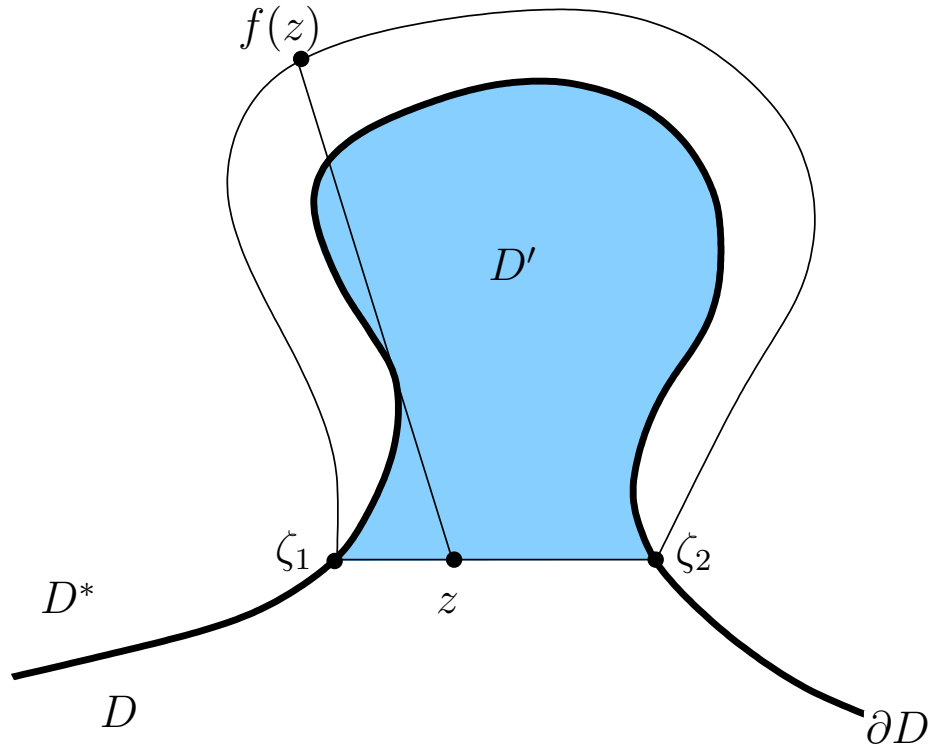
$$w = R_D(z) = \begin{cases} \gamma^*[\infty, \xi] \cap \gamma^*[\zeta^1, \zeta^2], & \text{for } z \in D \\ z, & \text{for } z \in \partial D \\ \infty, & \text{for } z = \infty. \end{cases}$$

Reflections

Suppose that ∂D admits a reflection $f : D \to D^*$ with

$$|f(z) - \zeta| \leq L|z - \zeta|, \ \forall z \in D, \zeta \in \partial D.$$

Proof:



Remember: $|f(z) - \zeta_i| \leq L|z - \zeta_i|$

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- the geometry of John disks *can* be compared with the outside (reflections), but only by going one way

Have a better understanding of why John disks look the way they do.