REMARKS ON CONFORMAL AND QUASICONFORMAL MAPS ONTO JOHN DOMAINS

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Abstract. Some properties of John disks and domains are considered. John disks are characterized in terms of a bound for hyperbolic distance, involving a one sided version of the Apollonian metric. Certain distortion properties of hyperbolic bilipschitz maps onto John disks are also studied. Finally it is shown that certain cone arcs in John domains in space satisfy a bounded turning condition.

1. Introduction

The purpose of this note is to present some properties of quasiconformal maps onto John domains. In Sections 2 and 3 I restrict the attention to *John disks*, that is planar, simply connected John domains. In Section 2 I characterize John disks in terms of a bound for hyperbolic distance, involving an internal distance version of the Apollonian metric. In Section 3 quasiconformal maps onto John disks and extensions of quasisymmetric boundary maps are studied. In Section 4 I look at bounded John domains D in $\mathbb{R}^n, n \geq 2$, for which there is a quasiconformal map of the unit ball $\mathbb{B}^n$ onto D, and find that images of radii are curves of bounded turning in D.

Notation and definitions. Let $n \geq 2$. Denote balls, circles, the unit disk and upper half plane as

$$B(x,r) = \{y \in \mathbb{R}^n \colon |x-y| < r\}, \quad S(x,r) = \partial B(x,r),$$

$$\mathbb{D} = \{z \in \mathbb{R}^2 \colon |z| < 1\}, \quad \mathbb{H} = \{z \in \mathbb{R}^2 \colon \operatorname{Im} z > 0\}.$$

If α is a rectifiable arc, $\ell(\alpha)$ denotes its length. Let $D \subset \mathbb{R}^n$ be a domain. Then internal length and diameter distances are defined by:

$$\lambda_D(x,y) = \inf\{\ell(\alpha) \colon \alpha \text{ rectifiable arc joining } x \text{ and } y \text{ in } D\},$$

$$\delta_D(x,y) = \inf\{\operatorname{diam}(\alpha) \colon \alpha \text{ arc joining } x \text{ and } y \text{ in } D\}.$$

If $D \subset \mathbb{R}^2$ is a Jordan domain, $\partial_r D$ denotes the set of rectifiably accessible boundary points of D, and $D_r = D \cup \partial_r D$. If D is a Jordan John disk (see below), then $\partial_r D = \partial D$. $\operatorname{dist}(x,A) = \inf_{y \in A} |x-y|$. Notation like $\operatorname{diam}_\lambda, \operatorname{dist}_\lambda$ etc. means that internal distance is used to measure diameters and distances. For an arc α, $\alpha[x,y]$ will denote the subarc from x to y. Closed and half open line segments from x to y are denoted by $[x,y]$, $[x,y)$ etc.

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If $D \subset \mathbb{R}^2$ is a simply connected proper subdomain, $h_D(z_1, z_2)$ will denote the hyperbolic distance in D.

A simply connected proper subdomain $D \subset \mathbb{R}^n$ is said to be an *a-John domain* if for any two points x_1 and x_2 in D there is a rectifiable arc $\alpha \subset D$ joining them with

$$\min_{j=1,2} \ell(\alpha[x_j, x]) \leq a \operatorname{dist}(x, \partial D)$$

for every x in α. This definition implies the existence of a point $x_0 \in \overline{D}$ such that every $x_1 \in D$ may be joined to x_0 by a rectifiable arc β with $\ell(\beta[x, x_1]) \leq b \operatorname{dist}(x, \partial D)$ for all $x \in \beta$. The point x_0 is called a *center* of D, and β is a *b-cone arc* joining x_1 to x_0. See [NV], [GHM]. If D is unbounded, $x_0 = \infty$.

A *John disk* is a simply connected John domain in the plane.

Finally note that one could have used *diameters* instead of lengths in the definitions of John domains above. See [NV, Section 2].

2. Hyperbolic distance and internal Apollonian metric

In this section we restrict ourselves to John disks. Absolute cross ratios in the euclidean and internal metrics are denoted by

$$|z_1, z_2, z_3, z_4| = \frac{|z_1 - z_3|}{|z_1 - z_4|} \frac{|z_2 - z_4|}{|z_2 - z_3|}, \quad |z_1, z_2, z_3, z_4|_\lambda = \frac{\lambda_D(z_1, z_3)}{\lambda_D(z_1, z_4)} \frac{\lambda_D(z_2, z_4)}{\lambda_D(z_2, z_3)},$$

respectively.

If D is a proper subdomain of $\overline{\mathbb{R}^2}$ we may define the *Apollonian metric* a_D by

$$a_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial D} \log \left(|z_1, z_2, w_1, w_2| \right), \ z_1, z_2 \in D. \tag{2.1}$$

See e.g. [Be] and [GH2]. From (2.1) we see that a_D is invariant under Möbius transformations.

Gehring and Hag [GH2] proved that a simply connected proper subdomain $D \subset \overline{\mathbb{R}^2}$ of hyperbolic type is a quasidisk if and only if there is a constant c, depending only on D, such that

$$h_D(z_1, z_2) \leq c\, a_D(z_1, z_2).$$

In the unit disk $\mathbb{D}$ we have $a_{\mathbb{D}}(z_1, z_2) = h_{\mathbb{D}}(z_1, z_2)$, [Be].

Next we introduce an *internal* Apollonian metric, a'_D, defined in terms of internal distances. Let $D \subset \mathbb{R}^2$ be a Jordan domain, and set

$$a'_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial_r D} \log \left(|z_1, z_2, w_1, w_2|_\lambda \right). \tag{2.2}$$

Remark 2.3. Note that a_D is only a *pseudometric* if ∂D is strictly contained in a circle C and D contains points which are symmetric with respect to C. Likewise it is possible that a'_D is only a pseudometric in special situations. Whenever D is the image of the unit disk under a quasimöbius homeomorphism (see below), however, it is easy to see from the proof of Lemma 2.8 that $a'_D(z_1, z_2) > 0$ whenever $z_1 \neq z_2$.

Remark 2.4. For every choice of $w_1, w_2 \in \partial_r D$ we have:

$$\frac{\lambda_D(z_1, w_j)}{\lambda_D(z_2, w_j)} \leq \frac{\lambda_D(z_1, z_2) + \lambda_D(z_2, w_j)}{\lambda_D(z_2, w_j)}$$
$$= \frac{\lambda_D(z_1, z_2)}{\lambda_D(z_2, w_j)} + 1 \leq \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)} + 1.$$

Thus by symmetry, and because w_1, w_2 are arbitrary,

$$a'_D(z_1, z_2) \leq \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right),$$

as in the euclidean case.

Let us define

$$j'_D(z_1, z_2) = \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right).$$

We see from Remark 2.4 that since j'_D is a metric (if D is simply connected), a'_D is always finite, even if (2.2) looks slightly alarming in that ∂D may contain non-rectifiably accessible points. Clearly,

$$a'_{\mathbb{D}}(z_1, z_2) = a_{\mathbb{D}}(z_1, z_2) = h_{\mathbb{D}}(z_1, z_2).$$

Besides, a'_D is invariant under maps $f\colon \overline{D} \to \overline{D'}$ preserving absolute cross ratios with respect to internal distances.

The metric j'_D was introduced by Langmeyer in [L]. See also [KL]. It is an *internal* version of the translation invariant j_D-metric introduced by Gehring and Osgood [GO]:

$$j_D(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)}\right).$$

Note that sometimes, as in [GO] and [Br2, Chapter 5], our j_D is multiplied by $1/2$. Langmeyer proved the following, [L, Theorem 3.1], [KL, Theorem 4.1].

Theorem 2.5 ([L], [KL])**.** *A simply connected domain $D \subset \mathbb{R}^2$ is a b-John disk if and only if there is a constant c such that*

$$h_D(z_1, z_2) \leq c\, j'_D(z_1, z_2) \tag{2.6}$$

for all pairs z_1, z_2 in D. The constants b and c depend only on each other.

This theorem is analogous to Corollary 1 in [GO].

Corollary 2.7. *Suppose that D is a simply connected Jordan domain and that there is a constant α such that*

$$h_D(z_1, z_2) \leq \alpha\, a'_D(z_1, z_2),$$

for all $z_1, z_2 \in D$. Then D is a b-John disk, with $b = b(\alpha)$.

Proof. Using Remark 2.4 we see that under the given assumptions there is a constant α such that

$$h_D(z_1, z_2) \leq \alpha\, a'_D(z_1, z_2) \leq \alpha\, j'_D(z_1, z_2).$$

Thus D is a b-John disk, $b = b(\alpha)$, by Theorem 2.5. □

If D is a John disk, we have the following bound for the hyperbolic distance.

Lemma 2.8. *Suppose that D is a Jordan c-John disk. Then there are constants α and β, depending only on c, such that*

$$h_D(z_1, z_2) \leq \alpha\, a'_D(z_1, z_2) + \beta.$$

In the proof *quasimöbius maps* will be used. Suppose $f\colon (X, \varrho) \to (Y, \sigma)$ is a homeomorphism between two metric spaces. Then f is said to be ϑ-quasimöbius if there is an increasing homeomorphism $\vartheta\colon [0, \infty) \to [0, \infty)$ such that for every quadruple $\{z_1, z_2, z_3, z_4\} \subset X$,

$$|f(z_1), f(z_2), f(z_3), f(z_4)|_\sigma \leq \vartheta(|z_1, z_2, z_3, z_4|_\varrho).$$

The homeomorphism f is said to be *η-quasisymmetric* if there is an increasing homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that

$$\frac{\sigma(f(z_1), f(z_2))}{\sigma(f(z_2), f(z_3))} \leq \eta\left(\frac{\varrho(z_1, z_2)}{\varrho(z_2, z_3)}\right)$$

for all $z_1, z_2, z_3 \in X$, $z_2 \neq z_3$.

Proof of Lemma 2.8. Suppose first that D is bounded and that $f\colon D \to \mathbb{D}$ is a conformal map. Then because D is a Jordan c-John disk we know from [NV, Section 7] and [Br1, Lemma 4.4] that the unique homeomorphic extension

$$f\colon (\overline{D}, \lambda_D) \to \overline{\mathbb{D}}$$

is η-quasisymmetric, with η depending only on c (Euclidean distance is used in $\overline{\mathbb{D}}$). By e. g. Theorem 11.3 in [H2] we may choose

$$\eta(t) = C \max\{t^\gamma, t^{1/\gamma}\},$$

where $C \geq 1$ and $\gamma \in (0, 1]$ depend on c only. By the proof of Theorem 3.2 in [V2] f is ϑ-quasimöbius with

$$\vartheta(t) = C^2 3^{2/\gamma} t^{3/2\gamma}$$

whenever $t \geq 1$. (We will only make use of ϑ when $t \geq 1$. Clearly

$$\sup_{w_1, w_2 \in \partial_r D} |z_1, z_2, w_1, w_2|_\lambda \geq 1.$$

For if not, interchange w_1, w_2 for any choice of such points.) By multiplying with an appropriate constant, if necessary, we may assume that $\vartheta(t) = Bt^\alpha$. Thus ϑ depends only on c. Since f is conformal in D we have

$$\begin{aligned} h_D(z_1, z_2) &= h_{\mathbb{D}}(f(z_1), f(z_2)) \\ &= a_{\mathbb{D}}(f(z_1), f(z_2)) \\ &= \log\left(|f(z_1), f(z_2), \omega_1, \omega_2|\right), \end{aligned}$$

where ω_1 and ω_2 are appropriate points in $\mathbb{T}$. Now using the fact that f is ϑ-quasimöbius, with $\vartheta(t) = Bt^\alpha$, and denoting $w_j = f^{-1}(\omega_j)$ we have

$$\begin{aligned}\log\left(|f(z_1), f(z_2), \omega_1, \omega_2|\right) &\le \log\left(\vartheta(|z_1, z_2, w_1, w_2|_\lambda)\right)\\ &= \log(B|z_1, z_2, w_1, w_2|_\lambda^\alpha)\\ &= \alpha \log(|z_1, z_2, w_1, w_2|_\lambda) + \log B\\ &\le \alpha a'_D(z_1, z_2) + \beta.\end{aligned}$$

Thus $h_D(z_1, z_2) \le \alpha\, a'_D(z_1, z_2)+\beta$. The same argument can be used in the unbounded case, starting with a quasisymmetric map from $\mathbb{H}$. □

It would be nice if we could remove the additive constant β. This is easy if $a'_D(z_1, z_2)$ is large. For if $a'_D(z_1, z_2) = t \ge 1$, then arguing as in [GH1, Lemma 4.3],

$$h_D(z_1, z_2) \le \alpha\, a'_D(z_1, z_2) + \beta = (\alpha + \frac{\beta}{t})\, a'_D(z_1, z_2) \le (\alpha + \beta)\, a'_D(z_1, z_2).$$

In any case we can characterize John disks in terms of a bound for hyperbolic distance with an additive constant.

Theorem 2.9. *A simply connected Jordan domain $D \subset \mathbb{R}^2$ is a b-John disk if and only if there are constants α and β such that*

$$h_D(z_1, z_2) \le \alpha\, a'_D(z_1, z_2) + \beta \tag{2.10}$$

for all pairs z_1, z_2 in D. Here α and β depend only on b, and b depends only on α and β.

Proof. If D is a John disk (2.10) follows from Lemma 2.8.

If (2.10) holds, Remark 2.4 gives

$$h_D(z_1, z_2) \le \alpha\, j'_D(z_1, z_2) + \beta.$$

Using this we show that (2.6) holds with $c = \max\{\alpha+\beta, 2\}$. Then D will be a b-John disk by Theorem 2.5.

We reason as in [GH1, Lemma 4.3]; see also [GH3, Section 9.2]. If $j'_D(z_1, z_2) = t \ge 1$, then

$$h_D(z_1, z_2) \le \alpha\, j'_D(z_1, z_2) + \beta = (\alpha + \frac{\beta}{t})\, j'_D(z_1, z_2) \le (\alpha + \beta)\, j'_D(z_1, z_2).$$

Assume that $\operatorname{dist}(z_2, \partial D) \ge \operatorname{dist}(z_1, \partial D)$. If $0 < t \le 1$ we have then

$$\left(\frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)} + 1\right)^2 \le e^{j'_D(z_1, z_2)} = e^t,$$

and

$$s = \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)} \le e^{t/2} - 1 < 1. \tag{2.11}$$

The inequality $|z_1 - z_2| \le \lambda_D(z_1, z_2)$ always holds, so that

$$z_1, z_2 \in D' = B(z_2, \operatorname{dist}(z_2, \partial D)) \subset D,$$

and thus $\lambda_D(z_1, z_2) = |z_1 - z_2|$. Now D' is a disk contained in D with center z_2. Because $\lambda_D(z_1, z_2)/\operatorname{dist}(z_2, \partial D)$ is invariant under affine maps, we may assume that $D' = \mathbb{D}$. Hence

$$h_D(z_1, z_2) \le h_{D'}(z_1, z_2) = \log\left(\frac{1+s}{1-s}\right).$$

Using (2.11) we have

$$\log\left(\frac{1+s}{1-s}\right) \le \log\left(\frac{e^{t/2}}{2-e^{t/2}}\right).$$

When $0 < t \le 1$ it is not very difficult to verify that

$$t + t^2 - \log\left(\frac{e^{t/2}}{2-e^{t/2}}\right) \ge 0.$$

Then

$$h_D(z_1, z_2) \le t + t^2 = t(1+t) < 2j'_D(z_1, z_2).$$

□

Conjecture 2.12. *A simply connected (Jordan) domain $D \subset \mathbb{R}^2$ is a b-John disk if and only if there is a constant c such that*

$$h_D(z_1, z_2) \le c\, a'_D(z_1, z_2)$$

for all z_1, z_2 in D.

3. Quasiconformal maps onto John disks

If $z = re^{it} \in \mathbb{D}$, let

$$B(re^{it}) = \{\varrho e^{i\vartheta} : r \le \varrho \le 1, |\vartheta - t| \le \pi(1-r)\}$$

for $0 \le r < 1$.

The following is an important characterization of (bounded) John disks.

Theorem 3.1 ([P])**.** *Let f map the unit disk $\mathbb{D}$ conformally onto a bounded domain $D \subset \mathbb{R}^2$. Then D is an a-John disk with center $z_0 = f(0)$ if and only if*

$$\operatorname{diam} f(B(z)) \le b \operatorname{dist}(f(z), \partial D) \tag{3.2}$$

for every z in $\mathbb{D}$. Here b depends on z_0 and a, while a depends only on b.

The statement as given here is the equivalence of (i) and (ii) in [P, Theorem 5.2]. See also Hag and Hag, [HH]. Let me introduce

$$I(re^{it}) = \mathbb{T} \cap \partial B(re^{it}),$$

see [P, Section 5.2]. As Pommerenke points out in [P, Exercise 5.2.4] one may replace condition (3.2) by

$$\operatorname{diam} f(I(z)) \le c \operatorname{dist}(f(z), \partial D) \tag{3.3}$$

where c is only dependent on b. It may be interesting to know that a lower bound holds, too, if D is a Jordan domain. In fact we can prove the following.

Theorem 3.4. *Let f map $\mathbb{D}$ conformally onto a bounded Jordan a-John disk $D \subset \mathbb{R}^2$ with center $z_0 = f(0)$. Then there is a constant c depending only on a and z_0 such that for every $z \in \mathbb{D}$ we have*

$$\frac{1}{c}\operatorname{dist}(f(z), \partial D) \le \operatorname{diam} f(I(z)) \le c \operatorname{dist}(f(z), \partial D). \tag{3.5}$$

The proof of Theorem 3.4 is included in the proof of a more general result which we state next.

Theorem 3.6. *Let $D \subset \mathbb{R}^2$ be a bounded Jordan domain and suppose that $f\colon \overline{\mathbb{D}} \to \overline{D}$ is a homeomorphism such that*

(a) *$f\colon \overline{\mathbb{D}} \to (\overline{D}, \lambda_D)$ is ν-quasisymmetric;*

(b) *$f\colon \mathbb{D} \to D$ is K-bilipschitz with respect to hyperbolic distances $h_{\mathbb{D}}$ and h_D in $\mathbb{D}$ and D, respectively.*

Then D is an a-John disk with center $z_0 = f(0)$. If $z \in \mathbb{D}$ and $J \subset \mathbb{T}$ is an arc with

$$\frac{1}{L} \le \frac{\operatorname{dist}(z, \mathbb{T})}{\ell(J)} \le \frac{\operatorname{dist}(z, J)}{\ell(J)} \le L, \tag{3.7}$$

we have

$$\frac{1}{M}\ell(J)|D_f(z)| \le \operatorname{diam} f(J) \le M\ell(J)|D_f(z)|, \tag{3.8}$$

where $|D_f(z)|$ denotes the operator norm of the derivative of f (which is quasiconformal in $\mathbb{D}$ by (a)).

In the above M depends only on ν and L, while a only depends on ν.

Before we embark on the proofs of Theorems 3.4 and 3.6, let us record a corollary to Theorem 3.6. Actually the statements in Theorem 3.6 and Corollary 3.9 were inspired by the relations (2.1) in the paper [R] of S. Rohde. These relations say that every quasisymmetric (with respect to euclidean distances) map of $\mathbb{T}$ into the plane has an extension to $\overline{\mathbb{D}}$ which satisfies the conclusions in Theorem 3.6. In this case D is a quasidisk.

Corollary 3.9. *Let D be a bounded Jordan domain. If $f\colon \mathbb{T} \to (\partial D, \lambda_D)$ is an η-quasisymmetric map, then f has an extension $f\colon \overline{\mathbb{D}} \to \overline{D}$ satisyfing (a), (b) and the conclusion in Theorem 3.6.*

Proof. First note that by Lemma 4.4 in [Br1] D is a c-John disk, with c depending only on η. See also [NV], Section 7. By Lemma 4.6 in [Br1] f has an extension $f\colon \overline{\mathbb{D}} \to \overline{D}$ such that (a) $f\colon \overline{\mathbb{D}} \to (\overline{D}, \lambda_D)$ is ν-quasisymmetric with ν depending only on η and (b) $f\colon \mathbb{D} \to D$ is K-bilipschitz with respect to hyperbolic distances in $\mathbb{D}$ and D. Now the result follows from Theorem 3.6. □

Part of the following argument (the upper bound in (3.11)) is adapted from "III ⇒ IV" of the proof of Theorem 3.1 in Heinonen's paper [H1]. The present situation, however, is simpler, since we have a homeomorphism that is quasisymmetric also on the boundary. I have included the entire argument here for completeness, and because my assumptions are slightly different from Heinonen's.

Proof of Theorem 3.6. Let $f\colon \overline{\mathbb{D}} \to \overline{D}$ be a homeomorphism satisfying (a) and (b). Then D is an a-John disk, with $a = a(\nu)$; [NV, Theorem 3.5].

Note also that each of the conditions (a) and (b) implies that f is quasiconformal in D. See e. g. [GH3, Section 7.7].

We first prove the upper bound in (3.8) under the assumptions (3.7). Be aware that I will write $f(z)$ for a point in $\overline{D}$, thus indicating also its preimage in $\overline{\mathbb{D}}$.

Let $z \in \mathbb{D}$ and $J \subset \mathbb{T}$ be a point and an arc satisfying (3.7). Take $w_0 \in \mathbb{T}$ such that $|f(z) - f(w_0)| = \operatorname{dist}(f(z), \partial D)$. Thus for all $f(w) \in [f(z), f(w_0)]$ we have $\lambda_D(f(z), f(w)) = |f(z) - f(w)|$. For all $\zeta \in \overline{\mathbb{D}}$ we have then by quasisymmetry:

$$\frac{|f(z) - f(\zeta)|}{|f(z) - f(w)|} \le \frac{\lambda_D(f(z), f(\zeta))}{\lambda_D(f(z), f(w))} \le \nu\left(\frac{|z-\zeta|}{|z-w|}\right). \tag{3.10}$$

In particular, if $\zeta_0 \in \overline{J}$ and $w = w_0$, (3.10) and (3.7) imply:

$$\begin{aligned} |f(z) - f(\zeta_0)| &\le \nu\left(\frac{|z-\zeta_0|}{|z-w_0|}\right)|f(z) - f(w_0)| \\ &\le \nu\left(\frac{|z-\zeta_0|}{\operatorname{dist}(z,\mathbb{T})}\right)\operatorname{dist}(f(z), \partial D). \end{aligned}$$

Remember that $\nu\colon [0,\infty) \to [0,\infty)$ is a non-decreasing homeomorphism. By (3.7), $|z - \zeta_0| \le \operatorname{dist}(z, J) + \ell(J) \le 2L \operatorname{dist}(z, \mathbb{T})$ which gives

$$|f(z) - f(\zeta_0)| \le \nu(2L)\operatorname{dist}(f(z), \partial D).$$

Finally taking $f(z_1), f(z_2) \in \overline{f(J)}$ with $\operatorname{diam} f(J) = |f(z_1) - f(z_2)|$, we have

$$\begin{aligned} \operatorname{diam} f(J) = |f(z_1) - f(z_2)| &\le |f(z_1) - f(z)| + |f(z) - f(z_2)| \\ &\le 2\nu(2L)\operatorname{dist}(f(z), \partial D). \end{aligned} \tag{3.11}$$

(Remember that $f(\zeta_0)$ was any point in $\overline{f(J)}$.)

Continuing the proof of the upper bound in (3.8), we see that because f is K-bilipschitz with respect to hyperbolic distances, we have as in (4.9) in [Br1] that

$$\frac{1}{4K}\frac{\operatorname{dist}(f(z), \partial D)}{\operatorname{dist}(z, \mathbb{T})} \le |D_f(z)| \le 4K\frac{\operatorname{dist}(f(z), \partial D)}{\operatorname{dist}(z, \mathbb{T})}, \tag{3.12}$$

for every z in $\mathbb{D}$. Using this with (3.11) and the assumptions (3.7) we have

$$\begin{aligned} \operatorname{diam} f(J) &\le 2\nu(2L)\operatorname{dist}(f(z), \partial D) \\ &\le 2\nu(2L)\, 4K \operatorname{dist}(z, \mathbb{T})|D_f(z)| \\ &\le 8\nu(2L)\, KL\, \ell(J)|D_f(z)|. \end{aligned}$$

This is the upper bound in (3.8) with $M = 8\nu(2L)KL$.

To prove the lower inequality in (3.8), note that using (3.12) and (3.7) again we have

$$\ell(J)|D_f(z)| \le 4K\ell(J)\frac{\operatorname{dist}(f(z), \partial D)}{\operatorname{dist}(z, \mathbb{T})} \le 4KL \operatorname{dist}(f(z), \partial D). \tag{3.13}$$

Furthermore, by choosing $w \in \overline{J}$ such that $\lambda_D(f(z), f(w)) = \operatorname{dist}_\lambda(f(z), f(J))$ we have for any $\zeta \in \overline{J}$

$$\begin{aligned} \operatorname{dist}(f(z), \partial D) &\le \operatorname{dist}(f(z), f(J)) \\ &\le \operatorname{dist}_\lambda(f(z), f(J)) \\ &= \lambda_D(f(z), f(w)) \\ &\le \nu\left(\frac{|z-w|}{|w-\zeta|}\right) \lambda_D(f(w), f(\zeta)). \end{aligned} \tag{3.14}$$

In particular this holds for some $\zeta \in \overline{J}$ such that $|\zeta - w| \ge \ell(J)/2\pi$. In addition we see, by picking $\xi \in \overline{J}$ with $|z-\xi| = \operatorname{dist}(z, J)$, that $|z-w| \le |z-\xi| + |\xi - w| \le (L+1)\ell(J)$. Using this in (3.14) we have

$$\begin{aligned} \operatorname{dist}(f(z), \partial D) &\le \nu(2\pi(L+1))\lambda_D(f(w), f(\zeta)) \\ &\le \nu(2\pi(L+1)) \operatorname{diam}_\lambda f(J). \end{aligned} \tag{3.15}$$

Finally, by e.g. [GNV, Theorem 5.14] (or [Br1, Lemma 3.4]) and [Br1, Lemma 3.6] we see that, if $f(z_1), f(z_2) \in \overline{f(J)}$ are such that $\operatorname{diam}_\lambda f(J) = \lambda_D(f(z_1), f(z_2))$, we have

$$\begin{aligned} \operatorname{diam}_\lambda f(J) = \lambda_D(f(z_1), f(z_2)) &\le \ell(\gamma[f(z_1), f(z_2)]) \\ &\le N_1 \operatorname{diam} \gamma[f(z_1), f(z_2)] \le N_1 N_2 \operatorname{diam} f(J), \end{aligned} \tag{3.16}$$

where N_1 and N_2 are appropriate constants from [GNV, Theorem 5.14] and [Br1, Lemma 3.6]; $\gamma[f(z_1), f(z_2)]$ is the hyperbolic line from $f(z_1)$ to $f(z_2)$ in D. By combining (3.13), (3.15) and (3.16) it follows that the lower inequality in (3.8) holds with $M = 4KL\nu(2\pi(L+1))N_1N_2$. Thus (3.8) holds with $M = \max\{8\nu(2L)KL,\ 4KL\nu(2\pi(L+1))N_1N_2\}$. □

Proof of Theorem 3.4. Let D be a bounded Jordan a-John disk with center z_0. Let $f\colon \mathbb{D} \to D$ be a conformal map with $f(0) = z_0$. Then f is a hyperbolic isometry, and by [NV, Section 7] and [Br1] $f\colon \overline{\mathbb{D}} \to (\overline{D}, \lambda_D)$ is ν-quasisymmetric for some ν depending only on a.

By taking $J = I(z)$, the upper bound in (3.5) now follows from (3.11).

The lower inequality in (3.5) can be verified by combining (3.15) and (3.16). □

4. Bounded turning arcs in John domains

In this section I will examine a possible generalization to higher dimensions of a result ([NV, Theorem 5.6]) of Näkki and Väisälä. The theorem alluded to says that in a John disk every hyperbolic *ray* γ from the center x_0 to the boundary satisfies a chord-arc condition, and in particular is of bounded turning. A rectifiable curve β satisfies a chord-arc condition if there is a constant b such that for any $x_1, x_2 \in \beta$

$$\ell(\beta[x_1, x_2]) \le b\,|x_1 - x_2|,$$

while it is of bounded turning if

$$\operatorname{diam} \beta[x_1, x_2] \le b\,|x_1 - x_2|.$$

See for example [JK], [LV, II.8.7].

Let $\mathbb{B} = \mathbb{B}^n = \{x \in \mathbb{R}^n \colon |x| < 1\}$, $n \geq 2$. In the following we will make the additional assumption that our domain D is the quasiconformal image of $\mathbb{B}$ (not necessarily under a global map, though). The reason for this is that there is no general mapping theorem in $\mathbb{R}^n$, $n \geq 3$, for quasiconformal maps.

Theorem 4.1. *Let $D \subset \mathbb{R}^n$ be a bounded a-John domain with center x_0. Assume that there exists a K-quasiconformal map $f \colon \mathbb{B} \to D$ with $f(0) = x_0$. Let $\beta = f([0, \xi))$, for any $\xi \in \partial\mathbb{B} = \mathbb{S}^{n-1}$. Let $x_1 = f(z_1)$ and $x_2 = f(z_2)$ be points of β with $0 \leq |z_1| < |z_2| < 1$. Then $\beta[x_0, x_2]$ is a b-cone arc joining x_2 to x_0 and*

$$\operatorname{diam} \beta[x_1, x_2] \leq b\, |x_1 - x_2|$$

for some $b = b(a, x_0, n, K)$.

Proof. We know from Theorem 3.10 in [NV] or Theorem 3.1 ("I $\Rightarrow$ III") in [H1] that the map $f \colon \mathbb{B} \to (D, \delta_D)$ is η-quasisymmetric, with η depending only on a, K and n. Let $\xi \in \mathbb{S}^{n-1}$, $\beta = f([0, \xi))$ and $x_1, x_2 \in \beta$ such that $0 \leq |z_1| < |z_2| < 1$, $z_j = f^{-1}(x_j)$.

Consider $G = f(B(z_1, |z_1 - z_2|))$. Then we see, as in the proof of Theorem 5.6 in [NV], that G is an a'-John domain, with a' depending on the data (a, x_0, K, n). Now $f|_{B(z_1, |z_1 - z_2|)}$ is K-quasiconformal, and thus $f \colon B(z_1, |z_1 - z_2|) \to (G, \delta_G)$ is ν-quasisymmetric, with ν depending only on a' and K, by appealing to [NV, Theorem 3.10] once more. Let us now follow the argument of "X $\Rightarrow$ I" in the proof of Theorem 3.1 in [H1]. Let $y \in S(z_1, |z_1 - z_2|)$ be such that

$$|f(y) - x_1| = \min_{|w - z_1| = |z_1 - z_2|} |f(w) - x_1|. \tag{4.2}$$

Then for all $\zeta \in [z_1, z_2]$ we have $|z_1 - \zeta| \leq |z_1 - y|$. Because f is quasisymmetric we have

$$\delta_G(f(\zeta), x_1) \leq \nu(1)\, \delta_G(f(y), x_1).$$

By (4.2) we have $\delta_G(f(y), x_1) = |f(y) - x_1| = \operatorname{dist}(x_1, \partial G)$. Take an $f(\zeta) \in \beta[x_1, x_2)$ with $\operatorname{diam} \beta[x_1, x_2] \leq 2|f(\zeta) - x_1|$. Then

$$\operatorname{diam} \beta[x_1, x_2] \leq 2\delta_G(f(\zeta), x_1) \leq 2\nu(1) \operatorname{dist}(x_1, \partial G). \tag{4.3}$$

Thus $\operatorname{diam} \beta[x_1, x_2] \leq 2\nu(1) \operatorname{dist}(x_1, \partial G) \leq 2\nu(1) \operatorname{dist}(x_1, \partial D)$, and β is a b-cone arc in D (with respect to diameter; see Section 1), with $b = 2\nu(1)$, joining x_2 to x_0. By (4.3) we also have that

$$\operatorname{diam} \beta[x_1, x_2] \leq 2\nu(1) \operatorname{dist}(x_1, \partial G) \leq b|x_1 - x_2|, \;\; b = 2\nu(1).$$

□

Corollary 4.4. *Let $D \subset \mathbb{R}^n$ be a bounded a-John domain with center x_0. Assume that there exists a K-quasiconformal map $f \colon \mathbb{B} \to D$ with $f(0) = x_0$. Let x_1, x_2 lie in the image β of a ray from 0 to $\partial\mathbb{B}$. Then*

$$\ell(\gamma) \leq c\, |x_1 - x_2| \tag{4.5}$$

for a quasihyperbolic segment γ from x_1 to x_2 in D. The constant c depends only on a, x_0, n and K. In particular we have

$$\ell(\gamma) \leq c' \operatorname{dist}(x_1, \partial D)$$

if $|z_1| < |z_2|$, $z_j = f^{-1}(x_j)$, with $c' = c'(a, x_0, n, K)$.

Proof. By Theorem 4.1 we have $\operatorname{diam}\beta[x_1, x_2] \leq b\,|x_1 - x_2|$, and using Theorem 3.4 in [V3] we see that

$$\lambda_D(x_1, x_2) \leq M\,\delta_D(x_1, x_2) \leq Mb\,|x_1 - x_2|,$$

where $M = M(a, n)$. Because D is the quasiconformal image of a uniform domain, Theorem 1.1 in [HR] (the Gehring-Hayman theorem for quasihyperbolic geodesics in quasiconformal images of uniform domains) states that there is a constant L depending on a, n and K only, such that $\ell(\gamma) \leq L\,\lambda_D(x_1, x_2)$ for the quasihyperbolic segment γ from x_1 to x_2. This proves (4.5) with $c = MLb$.

To prove the second statement, note that x_1 and x_2 lie in the cone arc β with x_1 between x_0 and x_2. Thus we have $|x_1 - x_2| \leq \operatorname{diam}\beta[x_1, x_2] \leq b\operatorname{dist}(x_1, \partial D)$. The result follows by using (4.5); $c' = MLb^2$. □

Remark 4.6. The last statement in Corollary 4.4 is interesting because it is not yet known if one can always use quasihyperbolic geodesics as cone arcs in John domains that are quasiconformal images of uniform domains or balls.

Note that there is in general no guarantee that the image of a radius in the ball is a quasihyperbolic ray.

Because every John disk is a *conformal* image of the unit disk, and because hyperbolic rays are conformal images of radii, we immediately obtain the result of Näkki and Väisälä [NV] that we referred to at the beginning of this section.

Corollary 4.7 ([NV], Theorem 5.6)**.** *Let D be a bounded a-John disk with center x_0. Then hyperbolic rays γ from x_0 to ∂D are cone arcs, and for any two points x_1 and x_2 in such rays γ we have*

$$\ell(\gamma[x_1, x_2]) \leq b\,|x_1 - x_2|$$

where $b = b(a, x_0)$.

A similar result holds for unbounded John disks [NV, Theorem 5.6].

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