— **MA8202 , hoesten 06** —

## More TASKS.

**BASIC TASKS.**
9.2 (p.69), 9.8 (p.70), 10.6, 10.7 (p.77), 10.10 (p.78).

**Task B1.** Let $A$ be an integrally closed domain, $P \in \operatorname{Spec} A$. Prove that $A_P$ is integrally closed.

**Task B2.** Prove that the intersection of a set of integrally closed subrings $R_i$ in a field $K$ (where $K$ is a fraction field for each $R_i$) is integrally closed.

**Task B3.** Let $K = \mathbf{k}(x)$ be a field of rational functions over a field $\mathbf{k}$ and $h \in \mathbf{k}[x]$ an irreducible polynomial. Notice that for any non-zero $f \in K$ one can write

$$f = h^n \frac{a}{b}$$

where $a$, $b$ are polynomials mutually prime with $h$. Show that the correspondence $f \to n$ defines a valuation on $K$.

**Task B4.** If $A$ is a Dedekind ring, then $A_S$ is also a Dedekind ring.

**Task B5.** Let $m \in M$. Prove that $\bar{m} = (m/1) \in M_S$ is non-zero if and only if $\operatorname{Ann}(m) \cap S = \emptyset$.

**Task B6.** Let $M$ a finite $A$-module and $S \subset A$ a multiplicative subset. Then $M_S = 0$ if and only if $\exists t \in S$ such that $s \in \operatorname{Ann} M$.

**MORE INVOLVED TASKS.**
9.1, 9.3, 9.6 (p.69), 11.6, 11.7 (p.86).

**Task M1.** Prove that there is an irreducible non-prime element in $\mathbb{Z}[\sqrt{10}]$.

**Task M2.** Let $K = \mathbf{k}(x)$ be a field of rational functions over a field $\mathbf{k}$. For

$$f = \frac{a}{b}$$

where $a$, $b$ are polynomials define $v(f) = \deg b - \deg a$. Check that this makes a valuation on $K$.

**Task M3.** Let $A$, $B$, $C$ ideals in a Dedekind ring. Show that

$$A \cap (B + C) = (A \cap B) + (A \cap C) \quad \text{and} \quad A + (B \cap C) = (A + B) \cap (A + C).$$

**Task M4.** Let $A$ has no zero-divisors. Then Spec $A$ is an irreducible topological space.

**Task M5.** Let $I$ an ideal in $A$ and $M$ an $A$-module. Prove that $M \otimes_A (A/I) = M/IM$.

**Task M6.** Let $A$ is a subring of $B$ and $S \subset A$ a multiplicative subset. Then $B_S = B$ if and only if any $t \in S$ is invertible in $B$.

**ADVANCED TASKS.**
9.7 (p.69), 10.8* (p.77), 11.5*, 11.10 (p.86).

**Task A1.** Let $R$ be a Noetherian local ring whose maximal ideal is principal and $\dim R > 0$. Show that $R$ is a domain.
(Hint: consider ideals $\mathrm{Ann}(x^{n-1}) \subset \mathrm{Ann}(x^n) \dots$ in $R$.)

**Task A2.** Show that the topological space $X = \mathrm{Spec}\ A$ is *quasi-compact*, that means if $X$ is equal to a union of a collection of open subsets, it is equal to a union of a finite subcollection of these open subsets.

**Task A3.** Let $A$ a local ring and $M$, $N$ finite $A$-modules. Then $M \otimes_A N = 0$ implies that either $M = 0$ or $N = 0$.

**Task A4.** Show that a homorphism $\varphi : M \to N$ of $A$-modules is a monomorphism iff for every maximal ideal $P \subset A$ the homorphism $\varphi_{[P]} : M_P \to N_P$ is a monomorphism.