— **MA8202 , hoesten 06** —

More TASKS.

BASIC TASKS.
9.2 (p.69), 9.8 (p.70), 10.6, 10.7 (p.77), 10.10 (p.78).

Task B1. Let A be an integrally closed domain, $P \in \operatorname{Spec} A$. Prove that A_P is integrally closed.

Task B2. Prove that the intersection of a set of integrally closed subrings R_i in a field K (where K is a fraction field for each R_i) is integrally closed.

Task B3. Let $K = \mathbf{k}(x)$ be a field of rational functions over a field $\mathbf{k}$ and $h \in \mathbf{k}[x]$ an irreducible polynomial. Notice that for any non-zero $f \in K$ one can write

$$f = h^n \frac{a}{b}$$

where a, b are polynomials mutually prime with h. Show that the correspondence $f \to n$ defines a valuation on K.

Task B4. If A is a Dedekind ring, then A_S is also a Dedekind ring.

Task B5. Let $m \in M$. Prove that $\bar{m} = (m/1) \in M_S$ is non-zero if and only if $\operatorname{Ann}(m) \cap S = \emptyset$.

Task B6. Let M a finite A-module and $S \subset A$ a multiplicative subset. Then $M_S = 0$ if and only if $\exists t \in S$ such that $s \in \operatorname{Ann} M$.

MORE INVOLVED TASKS.
9.1, 9.3, 9.6 (p.69), 11.6, 11.7 (p.86).

Task M1. Prove that there is an irreducible non-prime element in $\mathbb{Z}[\sqrt{10}]$.

Task M2. Let $K = \mathbf{k}(x)$ be a field of rational functions over a field $\mathbf{k}$. For

$$f = \frac{a}{b}$$

where a, b are polynomials define $v(f) = \deg b - \deg a$. Check that this makes a valuation on K.

Task M3. Let A, B, C ideals in a Dedekind ring. Show that

$$A \cap (B + C) = (A \cap B) + (A \cap C) \quad \text{and} \quad A + (B \cap C) = (A + B) \cap (A + C).$$

Task M4. Let A has no zero-divisors. Then Spec A is an irreducible topological space.

Task M5. Let I an ideal in A and M an A-module. Prove that $M \otimes_A (A/I) = M/IM$.

Task M6. Let A is a subring of B and $S \subset A$ a multiplicative subset. Then $B_S = B$ if and only if any $t \in S$ is invertible in B.

ADVANCED TASKS.
9.7 (p.69), 10.8* (p.77), 11.5*, 11.10 (p.86).

Task A1. Let R be a Noetherian local ring whose maximal ideal is principal and $\dim R > 0$. Show that R is a domain.
(Hint: consider ideals $\mathrm{Ann}(x^{n-1}) \subset \mathrm{Ann}(x^n) \dots$ in R.)

Task A2. Show that the topological space $X = \mathrm{Spec}\ A$ is *quasi-compact*, that means if X is equal to a union of a collection of open subsets, it is equal to a union of a finite subcollection of these open subsets.

Task A3. Let A a local ring and M, N finite A-modules. Then $M \otimes_A N = 0$ implies that either $M = 0$ or $N = 0$.

Task A4. Show that a homorphism $\varphi : M \to N$ of A-modules is a monomorphism iff for every maximal ideal $P \subset A$ the homorphism $\varphi_{[P]} : M_P \to N_P$ is a monomorphism.