Math 171: Project on Fourier Series

1 Overview

The aim of this assignment is to give you practice writing mathematical prose that is both readable and precise.

The subject of the project is Fourier series. The central part of the project is to prove Fejér's Theorem. This theorem deals with the question of uniform convergence of the Fourier series of a function. The statement of the theorem can be found below, together with some details of the mathematics involved.

There are several rounds of revision for this assignment. The first is to give a copy of your assignment to another student in the class. Part of your grade will be the quality of feedback you give to your partner in this phase. The papers will be then turned in a first time to me, at which point comments and suggestions for improvement will be made. After the papers are returned, you will be required to resubmit *edited or rewritten* versions. Editing is an integral part of the writing process.

Important dates:

Tuesday 11th of May. First draft given to peer in class for review.
Thursday 13th of May. Comments returned to peer.
Tuesday 18th of May. Complete first draft due to me.
Tuesday 25th of May. Assignments returned with comments.
Tuesday 1st of June. Final rewritten versions due.

Note: When submitting the final version, you must also submit your first draft, your peer review, your partner's review of your paper, and the tutor's comments on your first draft. Therefore, it is vital that you keep copies of everything.

2 Parts of the paper

Your paper should consist of the following sections:

- An Introduction: an ideal introduction gives a clear, concise, and accurate overview of the paper, presenting enough information to interest a reader in the rest of the work and persuade them of its importance without presenting so much that the reader is overwhelmed.
- The Proof: an ideal section containing the proof of a major theorem proves the theorem in a correct, logical way; it provides any information that the reader is unlikely to already know (or should be reminded of); and is laid out in a manner such that the reader always knows which step of the proof is being done, how it fits into the whole proof, and its relative importance in the proof.
- An Application: an ideal section about an application explains the application concisely including any context that may be useful for the reader to know; it demonstrates clearly the relevance of the application to the main topic; and does all this with just enough detail to convince the reader without so much that the reader is confused.

3 Guidelines for Writing

Your aim should be to make your paper understandable. You should imagine that another student in the class has asked for your help on this topic.

Use full sentences. The only sentence fragments that are acceptable are headings such as "Theorem" and "Proof". You should not use symbols such as $\therefore$, $\implies$, $\iff$, $\because$, nor should you use

abbreviations such as "s.t." or "iff". These detract from the clarity. It should be possible for someone with basic mathematical knowledge to read your paper aloud without hesitating over symbols or poorly constructed sentences.

Different parts of the paper have different purposes and these lead to different styles. In the first part, the primary aim is to interest a person in your paper, to give them an overview, but not to burden them with minutiæ. In the second part, the aim is to present the theorem and its proof. In this part, accuracy is paramount *but it is not possible to be accurate without being clear.* In the third part – the application – clarity is again foremost.

When proving the theorem, although the proof is in the book you should not regard this as a copying exercise. The style adopted in the book is not the best style for this type of paper.

4 Fejér's Theorem

In order to understand the statement of the theorem, we need some additional concepts. One we have already met, two we will meet soon, and one we shan't cover in class. You may use all of these concepts without proof, together with usual properties of continuous functions from $f : [a, b] \to \mathbb{R}$ and usual properties of their integrals.

Definition 4.1 *Let (V, d) be a vector space with a metric. Let (s_n) be a sequence in (V, d). Define the sequence (σ_n) by $\sigma_n := \frac{1}{n}(s_1 + \cdots + s_n)$. We say that (s_n) $(C, 1)$-converges to s if $(\sigma_n) \to s$. We extend this to series by saying that Σa_n $(C, 1)$-converges to a if the sequence of partial sums $(C, 1)$-converges to a.*

You will need the concept of *uniform continuity.* We shall encounter this for more general metric spaces later in the course, for this project we only need to know it for functions $f : [a, b] \to \mathbb{R}$.

Lemma 4.2 *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

In the usual $\epsilon - \delta$ definition of continuity, the choice of δ depends on both x and ϵ. What the above lemma says is that δ can be chosen independently of x (but it still depends on ϵ).

Lemma 4.3 *Let $C([-\pi, \pi], \mathbb{R})$ denote the space of continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$. This is a metric space with metric:*

$$d(f, g) = \sup\{|f(t) - g(t)| : t \in [-\pi, \pi]\}.$$

We write this as a lemma since it is not immediate that this is well-defined. However, for the proof of the theorem you may assume that this is so.

We note that $C([-\pi, \pi], \mathbb{R})$ is a vector space and thus the concept of series is well-defined. In the book, the notion of convergence in $C([-\pi, \pi], \mathbb{R})$ is called *uniform convergence* to differentiate it from various other ways of defining a metric on $C([-\pi, \pi], \mathbb{R})$.

Definition 4.4 *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function. The* Fourier series *of f is the series of continuous functions $a_0 + \sum(a_n \cos nt + b_n \sin nt)$ where:*

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt.$$

With this, we can state Fejér's theorem:

Theorem (Fejér's Theorem) *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function with $f(-\pi) = f(\pi)$. Then the Fourier series of f $(C, 1)$-converges to f in $C([-\pi, \pi], \mathbb{R})$.*

The proof of the theorem can be found starting on page 632 in the course text.

5 Rubric

The paper will be graded according to the scheme below. There are five categories: introduction, proof, application, overall presentation, and review and revision. In each section, there are four marks available, making a total of twenty. The specifics for each mark are laid out below. *This scale is incremental.* That means that to get, say, two marks on the introduction you have to not only satisfy the criteria laid out for two marks but also the criteria for one mark.

Mark	Criteria
Introduction	
1	Introduction is present.
2	Aims of the paper are stated. Broad area of mathematics to which material belongs is stated.
3	Aims of the paper are clearly stated. Concise background information is explained. Relevance of subject matter is stated.
4	A clear, concise, accurate introduction is present.
Proof of Theorem	
1	Methodological steps have been taken toward proving the result.
2	Theorem is correctly proven.
3	Relevant definitions are stated. Theorem is proved in a clear way with correct differentiation to the steps.
4	A complete, precise, concise and detailed proof is given. Where relevant, especially elegant or informative steps are highlighted.
Application	
1	An application of the subject matter is given.
2	The application is set in its proper context.
3	The relevance of the subject matter to the application has been explained.
4	The importance of this work to the application is evaluated.
Overall Presentation	
1	Paper is legible with adequate spelling, punctuation, and grammar.
2	Paper is neatly and clearly presented with good spelling, punctuation, and grammar.
3	Secondary sources are cited correctly.
4	Segue between sections is smooth and logical.
Review and Revision	
1	First draft, peer review, partner's peer review, and tutor's review are present.
2	First draft has all sections, peer review comments on all sections of partner's first draft (even if said section is missing).
3	Final version shows improvement in accordance with partner's review, where relevant. Peer review is conducted according to this rubric. Tutor's recommendations have been implemented.
4	Peer review gives practical guidelines for improvement in partner's paper.