Research Proposal

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Introduction

My current and planned research divides into three areas: the differential topology of loop spaces, the properties of two-vector bundles, and operations in higher K–theories. This research lies in the areas of differential topology, differential geometry, and algebric topology and it has further application to mathematical physics. In addition to the techniques of the areas already mentioned, my research has a heavy reliance on functional analysis.

I have been working on the first of the three areas since my Ph.D. whilst the latter two have been developed more recently. My research into the properties of two-vector bundles is in collaboration with Nils Baas and Tore Kro of NTNU. My study of operations in higher K–theories is in collaboration with Sarah Whitehouse of Sheffield University and is currently part of an EPSRC-funded project, GR/S76823/01.

The Differential Topology of Loop Spaces

Overview

The aims of my research in this area are twofold: to develop an index theory for loop spaces and to define a calculable semi-infinite de Rham cohomology. This research will help advance the study of loop spaces as smooth manifolds.

Often the most powerful theories in mathematics are those which bridge the gaps between different areas. The original index theory was one of these in that it provided a passage between algebraic topology on the one hand and geometry and mathematical physics on the other. This has produced many important theorems in all the areas that it links. Index theory for loop spaces has already proved itself extremely useful even though it is not yet a rigorous theory. Just by knowing the form it should take, mathematicians and physicists have been able to transfer ideas from one area to another. A full theory would enable us to transfer results as well. I have already made significant progress on this project: in [66] I showed how to construct the operators that are the fundamental objects of study in index theory and in [68] I will give a rigorous definition of the topological index of such an operator.

Semi-infinite de Rham cohomology has the potential to be of similar importance. Ordinary de Rham theory provides a bridge between algebraic topology and geometry and this should also hold for the semi-infinite variety. With its

links to Floer homology and loop groups, semi-infinite de Rham theory will also bring in such diverse topics as symplectic geometry, mathematical physics, representation theory, string topology and others. My thesis, [64], contained a basic definition of semi-infinite de Rham cohomology but there are many ways to vary this construction. Recent work in the field of Floer theory, my work on loop spaces, and the emergence of parametrised homotopy theory and generalised complex structures means that this is an apposite time to revisit this topic.

Background

Loop spaces occur throughout mathematics. They have always been fundamental objects in algebraic topology and functional analysis, though in markedly different ways. Recently, they have also moved to the centre-stage in differential topology, geometry, and mathematical physics. The intersection is the study of loop spaces as smooth manifolds and therefore this has the potential to be a rich theory with many applications to other areas of mathematics.

From the early days of geometry it has been convenient to view a space of loops in a manifold as a manifold in its own right, though this is not always a rigorous concept. The primary motivation for doing so was usually to gain some insight into the original manifold through studying particular paths. The proof of Bott periodicity using Morse theory is a classical example of this, see [51] for an exposition.

This changed in the 1980s with the rise of string theory. A central idea of string theory is that space is a loop space. Loop spaces thus moved from being auxiliary to primary objects. Again motivated by considerations from string theory, one purpose of the study of loop spaces is to repeat the successes of finite dimensional geometry in the infinite realm. Often in so doing new aspects come to light. For example, the theory of loop groups has proved a fertile ground for research with its links to twisted K–theory, [30, 31, 32, 33, 55].

There are usually difficulties in extending some technique or construction from finite dimensions to infinite. This is even the case with the definition of an infinite dimensional manifold, [41, 42, 45, 52, 54]. The difficulties often lie in differential topology or geometry and require techniques from functional analysis in their solution.

A prime example of this is the problem of doing index theory on loop spaces. Index theory is a highly successful theory in finite dimensions, providing links between algebraic topology, geometry, and mathematical physics; see [47, 62]. Thus information and insights can freely move between the subject areas. The key steps in index theory are the following.

1. The input is a differential operator $D : \Gamma(E) \to \Gamma(F)$ where E, F are vector bundles over a manifold. There is a technical condition that this operator needs to satisfy called *ellipticity*. Many problems in geometry and physics give rise to such an operator.

2. To this operator we assign a *topological index*. The method of doing this is to start with a class in some K–theory, called the *symbol* of the operator, which we push-forward to the coefficient ring. This results in an integer.

3. We also consider the *analytical index*: Index $D := \dim \ker D - \dim \operatorname{coker} D$. The aforementioned ellipticity condition is to ensure that this is defined.

4. The index theorem states that these two indices are the same.

Two footnotes to this are that there is often a simple cohomological formula for the topological index, and that in the presence of a group action there is an equivariant version of the above wherein the indices become characters of the group. Moreover, if there is a group action then the cohomological formula is in terms of the fixed point set.

In [76], Witten formally applied this to the Dirac operator on a loop space with the natural circle action. As the fixed point set of this action is the constant loops, which is just the original manifold, the fixed point form of the topological index is in terms of finite dimensional things. The resulting expression, now known as the Witten genus, can therefore be rigorously defined and has since been extensively studied; for example [2, 28, 38, 40, 48, 67, 70, 73]. It has proved to be an important object in many areas of mathematics. Many of its properties were conjectured from its original construction by analogy with the problem in finite dimensions but as that construction was purely formal the analogies did not lead to proofs. Other methods had to be developed to fill in the gaps. This, of course, led to new and interesting mathematics but a genuine index theory for loop spaces would provide a deeper understanding of the mathematics involved.

The problems in doing index theory on loop spaces are many. Firstly, there is the basic problem of defining the operator. In the case of the Dirac operator one can set up the ingredients necessary for its definition exactly as in finite dimensions but two steps in its definition in finite dimensions fail when extended to infinite dimensions. There have been many partial solutions proposed to this problem which work by exploiting a particular feature of some special space, for example [16, 63]. In [66], I found a solution which worked for all loop spaces. One of the strengths of this method is that none of the initial data is changed, it is just used in a slightly different way.

Secondly, there is the question as to which K–theory should hold the symbol of the operator. This question is further obscured by the plethora of properties that this symbol possesses. Thus one can consider "high-level" K–theories which preserve as much of this structure as possible, and "low-level" ones which preserve just as much as is needed to define a reasonable theory. There are several current proposals for the high-level K–theory: [71] and [5] being the two front-runners. At the low-level, I have considered the K–theory of equivariant, positive-energy Hilbert bundles and found it to be a usable theory. Specifically, it is non-trivial and has a fixed point theory. Using this theory I can give a rigorous construction of the Witten genus as the topological index of the Dirac operator. The details of this will appear in [68].

Thirdly, there is the analytical index. Although one could attempt a basic definition, many of the techniques used to study the index in finite dimensions rely on the fact that there are enough Hilbert spaces around to impose additional regularity when it is needed. These in turn depend for their definition on the existence of a measure on the manifold. This introduces considerable difficulties when working with loop spaces.

Finally, once one has defined all of the structure there is the proof of the index theorem itself to consider.

Another example of extending finite dimensional structure to infinite dimensions is Floer theory, [50, 60]. For loop spaces of certain manifolds, including symplectic, it is possible to identify a "middle dimension". When adapting a construction from finite to infinite dimensions one can often choose to use this "middle dimension" in a non-trivial way. The term "semi-infinite" is used to distinguish these from standard theories. One such theory is Floer homology, originally introduced to solve the Arnold conjecture in finite dimensional symplectic geometry. It can be viewed as a semi-infinite version of Morse theory. There are many different ways of constructing the cohomology of a manifold, each with its strengths and areas of application. The fact that they all compute the same thing means that they can be used to pass information from one area to another. Each method should have a semi-infinite version for loop spaces.

I am particularly interested in semi-infinite de Rham cohomology. In defining this one encounters similar problems to those encountered in defining the Dirac operator. Due to technical considerations one cannot use the same techniques for the solution. In my thesis, [64], I proposed a solution to this problem. When applying this theory to certain simple test cases I found that to get a calculable theory I had to use some additional structure particular to those cases which is not available for loop spaces. Specifically, I used either a measure or a filtration by finite dimensional submanifolds. Therefore there are still many questions as to the best construction of semi-infinite de Rham cohomology for loop spaces.

These two areas, index theory and semi-infinite theory, form the focus of my current research on loop spaces. In working in these areas I have had to use results and techniques from a broad range of mathematics; particularly functional analysis, differential topology, and geometry. Moreover, I have found that most occurrences of loop spaces can be linked to these areas and thus come within this remit. Thus while providing me with a definite focus to my work I also have the freedom to investigate and use techniques from many areas of mathematics.

Planned Research

1. Having defined the Dirac operator and its topological index, the next stage is to consider the analytical index. I shall start this investigation by looking at the space of smooth sections of a vector bundle over a loop space. This investigation will require a thorough understanding of what it means for a section to be "smooth". I intend to use the calculus of [42] in this. In order to find a reasonable topology on this space it may be necessary to limit the sections in some fashion.

2. Once the topological and analytical indices have been defined, the next step is to formulate an index theorem and develop a programme for its proof. The guide for this programme will be the proof of the index theorem in finite dimensions but it is anticipated that there will be many twists in fitting this guide to infinite dimensions. It is likely that these twists will themselves be of independent interest.

3. On another tack, I intend to examine the properties of the Dirac operator itself. One key property is its behaviour with regard to the action of the diffeomorphism group of the circle. In [66] I showed that the natural action of

this diffeomorphism group does not act on the operator but I also showed that this action could be modified so that it did. The natural action has been much studied and it will be interesting to see what carries over to the modified action.

4. The construction of the Dirac operator revealed considerable geometrical structure on loop spaces that had not hitherto been seen. One of the key features is an inner product on the cotangent spaces. In finite dimensions the easy transition between tangent and cotangent vectors is an essential part of geometry. I shall examine what from finite dimension geometry extends to the infinite.

5. The K–theory of equivariant Hilbert bundles is a "low-level" theory. There should be forgetful maps to this from the proposed "high-level" theories of [71] and [5]. I shall examine the structures of all three with a view to defining these maps and determining the precise relationships between these theories.

6. With regard to semi-infinite de Rham cohomology, the recent work of [1, 61, 74] has suggested a test case that I did not consider in my thesis: the loop space of a cotangent bundle. As it is a loop space, this case is more applicable to the general case than other test cases; as it is a vector bundle it may be more tractable to analysis than a generic loop space. An essential part of any successful semi-infinite de Rham theory is an infinite dimensional Thom isomorphism. This will involve an integration theory on the fibres of the cotangent bundle. I shall investigate how this may be done, starting by considering the work of Gel'fand and Vilenkin [34].

7. One unfortunate aspect of the semi-infinite structure of the loop space of a symplectic manifold is that it is based on a choice of compatible complex structure for the original symplectic manifold. Unless the manifold is Kähler this will not be integrable and thus the semi-infinite structure is not integrable. The solution to this will probably lie with the theory of *generalised complex structures*, introduced by Hitchin in [37] and continued by Gualtieri and others. See [8, 27, 35, 36, 49, 77, 78, 79] for a flavour of the research in this area. Specifically, there is an interesting relationship between the spinor space of the generalised complex bundle and the exterior power of the cotangent bundle.

8. One aspect of the theory of loop spaces that occurs throughout is the notion of *locality*. It is often desirable that some construction on a loop space depend locally on the original manifold. This can easily be achieved by ensuring that it depend *pointwise*, but this is a stronger notion than is often required. For example, the tangent bundle has pointwise dependence but the cotangent bundle only has local dependence. Although much can be done purely with objects with pointwise dependence, this approach does not reveal the full range of what is achievable. Whilst working on loop spaces I shall examine this issue to see where the boundary between pointwise and local objects lies.

9. The differential topology of loop spaces has recently developed very rapidly in many different directions. The basics of this theory can be unfamiliar to a researcher in topology as they involve quite intricate concepts from functional analysis. I have recently written seminar notes on the fundamentals of this subject, [65]. These are intended as an introduction to the differential topology of loop spaces for those familiar with ordinary differential topology but less

familiar with infinite dimensional calculus. I intend to continue to write these notes with a view to publishing them as a book.

10. Whilst occupied in the above research I shall look for ways to forge links with other topics in this area. In particular, the recent theory of string topology [19, 20, 25, 26] promises rich structure which should interact well with the objects that I study.

Two-Vector Bundles

Overview

The aim of my research in this area is to develop a theory of the differential geometry of two-vector bundles.

Ordinary vector bundles have a key role in joining many concepts in mathematics due to their intrinsic geometric nature. By extension, a geometric theory of two-vector bundles will forge new links between algebraic topology and geometry, and thence to mathematical physics. Strong evidence for the importance of two-vector bundles is provided by the pervasive theory of gerbes, of which two-vector bundles are a generalisation.

It is clear from the outset that one needs to use infinite dimensional geometry to develop the geometry of two-vector bundles. Therefore my experience with considering infinite dimensional manifolds will be of considerable use in this project. Also one of the main applications of the theory of two-vector bundles is to loop spaces and so again my expertise will prove invaluable.

This work is being undertaken in collaboration with Nils Baas and Tore Kro of NTNU.

Background

As I have already mentioned, index theory is a powerful theory which links algebraic topology with geometry and analysis. At its heart is K–theory. The strength of K–theory in this regard comes from the fact that it is an algebraic theory which is represented by geometric objects, namely vector bundles. Other examples of cohomology theories represented by geometric objects are de Rham cohomology and complex cobordism.

Any cohomology theory which is representable by geometric objects has the potential to be a useful bridge between geometry and algebraic topology. There is a ranking of cohomology theories called the *chromatic filtration* which, loosely, is a measurement of how much information the cohomology theory sees. Ordinary cohomology has chromatic filtration 0, K–theory has chromatic filtration 1, and complex cobordism has infinite chromatic filtration. Two-vector bundles were introduced in [5] as a candidate for the geometric object underlying a cohomology theory of chromatic filtration 2.

Together with Nils Baas and Tore Kro, I have begun a project to develop this theory further. My particular interest is to study the geometric properties of two-vector bundles, though we intend to study all aspects of the theory.

Two-vector bundles are closely related to gerbes. Indeed, a gerbe is to a two-vector bundle what a line bundle is to an ordinary vector bundle. There is

an extensive theory of gerbes already in the literature spanning a wide range of areas of mathematics: algebraic geometry, algebraic topology, differential topology, and geometry; [3, 4, 14, 15, 18, 46, 53]. We intend to adapt this as applicable to two-vector bundles.

Planned Research

1. My first goal is to define a connection on a two-vector bundle. The first step towards this is to pin down a geometrical definition of the fibre and total space of a two-vector bundle. This needs a "topologification" of the definition of [5]. Part of this study involves examining candidates for the fibres to determine the most suitable one.

2. Having settled on the fibres, I will consider how to define a connection on a two-vector bundle. As a guide to this I will consider the various methods of defining a connection on a gerbe.

3. I shall then determine how to use this connection to define characteristic classes. Guides for this will be Chern-Weil theory for ordinary bundles and the established theory for gerbes that yields the Dixmier-Douady class. This will involve the correct definition of the curvature of the connection.

4. The loop of a gerbe is a line bundle on the loop space, as explained in [17]. Therefore the loop of a two-vector bundle should be related to vector bundles on a loop space. I intend to investigate how this looping should work. It is likely that a crucial piece of this construction will be something analogous to holonomy.

5. The notion of a two-vector bundle can be generalised to a two-bundle. I shall generalise the definition of a topological two-vector and determine how to generalise the theory of two-bundles to this case.

6. Another candidate for an *elliptic object* is the theory of Stolz and Teichner, [71]. As part of the investigation I shall consider the links with their theory to see where the two overlap.

Stable and Unstable Operations in Higher K–Theories

Overview

My research in this area is currently as part of the EPSRC-funded project GR/S76823/01. The principal investigator is Sarah Whitehouse of Sheffield University.

The aim this project is to understand the relationship between stable and unstable phenomena as seen by the Morava K–theories.

Algebraic topology splits into two pieces: stable and unstable. The break is not clean, however, and many interesting problems require techniques from both pieces in their solution. Therefore it is of considerable interest to study the relationship between the two parts. Algebraic topologists use cohomology theories to convert topological problems into algebraic ones. Different cohomology theories "see" different parts of algebraic topology. Of particular importance

are the Morava K–theories, both for their high degree of calculability and their combined ability to "see" most of algebraic topology. Hence studying the interface between stable and unstable phenomena using the Morava K–theories is both a viable task and of considerable benefit to algebraic topology.

Background

Part of the structure that comes with a generalised cohomology theory are algebras of operations and co-operations. These contain considerable information about the theory. The general theory has been laid out in [9, 10].

In the 1970s, Morava introduced a family of cohomology theories now known as the Morava K–theories. For each prime p there is a sequence of theories $K(0), K(1), K(2), \ldots$. Each is periodic and all have nice properties which aid calculation. Their importance in stable homotopy theory has been well-established, [29, 39, 56, 57, 58].

One can associate to a cohomology theory E a *localisation* functor, L_E, on the stable homotopy category. This is essentially a projection onto the part of the stable homotopy category that this cohomology theory "sees". In general localisation is quite complicated but for the Morava K–theories (at least for $n \geq 1$), work of Bousfield [11] (for $n = 1$) and Kuhn [44] (for all $n \geq 1$) has given a surprising description of $K(n)$–localisation, namely that it factors through the zeroth space functor, Ω^∞. In other words, they have constructed a functor Φ_n from the homotopy category of p–local based spaces to the homotopy category of $K(n)$–local spectra such that $L_{K(n)} = \Phi_n \Omega^\infty$. This has many interesting consequences which have been often exploited, for example in [12, 13, 43, 59]. One consequence is that the natural map from stable operations to unstable operations is split for the Morava K–theories.

The aim of our research is to study the algebras of stable and unstable operations for the Morava K–theories using the Bousfield-Kuhn functor and the algebraic descriptions of the co-operations in [9, 10, 75]. We have recently submitted our first result which gave a simple topological description for this splitting map, [69].

As already mentioned, this work is being carried out as part of an EPSRC-funded project. Once my association with this project has finished, I shall continue to carry out research in this area in collaboration with Sarah Whitehouse. As well as continuing with the broad aims of the current research project, I intend to look at the more topological aspects of this area. Much of the current study of Morava K–theories uses their algebraic properties and descriptions but their original definition was firmly based on the geometric ideas of Baas and Sullivan [6, 7, 72]. The result in [69] strongly suggests that this will prove a fruitful area of research.

Planned Research

1. Within the bounds of the EPSRC-funded project "Stable and Unstable Operations in Higher K–Theories", my research will continue along the following lines as laid out by Sarah Whitehouse.

Using the Bousfield-Kuhn functor and the description of the splitting in [69] we shall study the relationship between stable and unstable operations in the

Morava K–theories. The aim is a description analogous to that of Bousfield wherein a method is given for p–adic K–theory to generate unstable operations from their stable counterparts.

We shall then extend the above to the Morava K–theory of a spectrum in terms of the Morava K–theory of its zeroth space. This is also done by Bousfield for p–adic K–theory and is related to work of Rezk [59].

The Morava K–theory $K(1)$ is closely related to ordinary K–theory mod p. The algebras of stable and unstable operations for ordinary K–theory have concrete algebraic descriptions which have also been studied in the p–local case, [21, 22, 23, 24]. We shall also relate this work to these descriptions.

The prime $p = 2$ has some subtleties not present at other primes. It is often convenient work at odd primes to avoid difficulties. We shall examine the case of $p = 2$ to see which of our results remains valid.

2. Outside the remit of the EPSRC-funded project, I shall examine the interaction between geometry and algebra in the Morava K–theories. The initial aim of this is to derive topological explanations for known algebraic phenomena.

Funding and Projects

I am keen to seek funding for several aspects of my research. In particular I would look for funding for a project entitled "Index Theory on Loop Spaces" based on the first part of my research plan. One of the aims of this project would be to establish and develop collaborations with researchers in my own and other institutions. There are many mathematicians interested in this subject due to the broad range of mathematics that it involves and so I anticipate great interest in such a project. Also due to the range of techniques needed in the study, there are many mathematicians who could usefully contribute to this study whether or not they are currently engaged with loop spaces.

There are also several aspects of the study of loop spaces that would make good doctoral research projects. From my planned research, the study of the properties of the Dirac operator would be suitable, as would the semi-infinite structure of the loop space of cotangent bundles. Both of these are important problems which are finite in scope but which could lead to other more open questions.

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