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Fourier Decompositions of Loop Bundles

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Abstract. In this paper we investigate bundles whose structure group is the loop group $LU(n)$. These bundles are classified by maps to the loop space of the classifying space, $LBU(n)$. Our main result is to give a necessary and sufficient criterion for there to exist a Fourier type decomposition of such a bundle ξ. This is essentially a decomposition of ξ as $\zeta \otimes L\mathbb{C}$, where ζ is a finite dimensional subbundle of ξ and $L\mathbb{C}$ is the space of functions, $C^\infty(S^1, \mathbb{C})$. The criterion is a reduction of the structure group to the finite rank unitary group $U(n)$ viewed as the subgroup of $LU(n)$ consisting of constant loops. Next we study the case where one starts with an n dimensional bundle $\zeta \to M$ classified by a map $f : M \to BU(n)$ from which one constructs a loop bundle $L\zeta \to LM$ classified by $Lf : LM \to LBU(n)$. The tangent bundle of LM is such a bundle. We then show how to twist such a bundle by elements of the automorphism group of the pull back of ζ over LM via the map $LM \to M$ that evaluates a loop at a basepoint. Given a connection on ζ, we view the associated parallel transport operator as an element of this gauge group and show that twisting the loop bundle by such an operator satisfies the criterion and admits a Fourier decomposition.

1. Introduction

In this paper we study infinite dimensional vector bundles whose structure group is the loop group, $LU(n)$. We call such a bundle a "rank n loop bundle". Here $U(n)$ is the unitary group of rank n and for any finite dimensional smooth manifold M, LM denotes the space of smooth loops, $LM = C^\infty(S^1, M)$. A rank n loop bundle will have fibers isomorphic to the loop space $L\mathbb{C}^n$. Now $\mathbb{C}^n$-valued functions on the circle have Fourier expansions and one of our goals in this paper is to give necessary and sufficient conditions for there to exist such an expansion fiberwise on a loop bundle.

The Fourier expansion of elements in $L\mathbb{C}^n$ can be viewed as a map

$$\Phi : L\mathbb{C}^n \longrightarrow \mathbb{C}[\![z, z^{-1}]\!] \otimes \mathbb{C}^n$$

where $\mathbb{C}[\![z, z^{-1}]\!]$ denotes the ring of formal power series in a variable z and its inverse. Let $\mathbb{C}[\![z]\!] \subset \mathbb{C}[\![z, z^{-1}]\!]$ be the subalgebra consisting of power series that only involve non-negative powers of z. Recall that the inverse image, $\Phi^{-1}(\mathbb{C}[\![z]\!] \otimes \mathbb{C}^n)$,

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which we call $L_+\mathbb{C}^n$, consists of boundary values of holomorphic maps $f : D^2 \to \mathbb{C}^n$. Notice that $L_+\mathbb{C}^n \subset L\mathbb{C}^n$ has the following properties:

(1) Consider the inner product on $L\mathbb{C}^n$ defined by

$$(\alpha, \beta) = \int_{S^1} \langle \alpha(t), \beta(t) \rangle dt. \tag{1.1}$$

Then $L_+\mathbb{C}^n \subset L\mathbb{C}^n$ is an infinite dimensional, infinite codimensional subspace that is closed with respect to the topology induced by the inner product. Moreover, there is an orthogonal decomposition

$$L\mathbb{C}^n = L_+\mathbb{C}^n \oplus L_-\mathbb{C}^n$$

where the space $L_-\mathbb{C}^n$ consists of loops whose Fourier series only involve negative powers of z.

(2) $L_+\mathbb{C}^n$ is invariant under multiplication by z:

$$zL_+\mathbb{C}^n \subset L_+\mathbb{C}^n.$$

Here we are identifying the Laurent polynomial ring $\mathbb{C}[z, z^{-1}]$ as a subalgebra of $L\mathbb{C}$, and hence $L\mathbb{C}^n$ is a module over $\mathbb{C}[z, z^{-1}]$. Furthermore, the codimension of $zL_+\mathbb{C}^n$ in $L_+\mathbb{C}^n$ is n.

Notice that there is a filtration of subspaces,

$$\cdots \subset z^{-k}L_+\mathbb{C}^n \subset z^{-(k+1)}L_+\mathbb{C}^n \subset \cdots \subset L\mathbb{C}^n$$

where the union $\bigcup_k z^{-k}L_+\mathbb{C}^n$ is a dense subspace of $L\mathbb{C}^n$. We think of this filtration as the "Fourier decomposition" of $L\mathbb{C}^n$ and will define the existence of a "Fourier decomposition" of a loop bundle ξ as the existence of a subbundle, $\xi_+ \subset \xi$, which satisfies the above two properties in a fiberwise manner.

One of the main goals of this paper is to prove the following theorem. It can be viewed as saying that Fourier decompositions of loop bundles are most often not possible.

THEOREM 1.1. *Let $\xi \to X$ be a rank n loop bundle. Then ξ has a Fourier decomposition if and only if the structure group of ξ can be reduced to $U(n)$ viewed as the subgroup of the constant loops in $LU(n)$.*

Notice that since the classifying space of the infinite dimensional group $LU(n)$ is the loop space of the classifying space of $U(n)$, $LBU(n)$, a rank n loop bundle over a space X is classified by a map $f : X \to LBU(n)$. We call the "underlying n-dimensional bundle", ξ_U, the bundle classified by the composition

$$X \xrightarrow{f} LBU(n) \xrightarrow{e} BU(n)$$

where $e : LY \to Y$ is the map that evaluates a loop at $1 \in S^1$. Let D^2 be the two dimensional disk. Restriction to the boundary defines a fibration between the mapping spaces $\rho : Map(D^2, BU(n)) \to LBU(n)$.

The following is essentially a restatement of the above theorem, but it is quite useful in practice.

COROLLARY 1.2. *Let $f : X \to LBU(n)$ classify a loop bundle $\xi \to X$. Then ξ has a Fourier decomposition if and only if there is a lift of f to $Map(D^2, BU(n))$.*

The following is an immediate corollary of theorem 1.1 and gives a description of any loop bundle that has a Fourier decomposition.

COROLLARY 1.3. *Let $\xi \to X$ be a rank n loop bundle and let $\xi_U \to X$ be its underlying n-dimensional bundle. Then ξ has a Fourier decomposition if and only if there is an isomorphism of loop bundles*

$$\xi \cong L\mathbb{C} \otimes \xi_U.$$

Let M be a simply connected manifold and $\zeta \to M$ an n-dimensional complex bundle classified by a map $f_\zeta : M \to BU(n)$. Consider the induced rank n loop bundle over the loop space, $L\zeta \to LM$.

Let $\gamma \in LM$. The fiber over γ of $L\zeta$ is the space of sections of the pull back of ζ over the circle,

$$L\zeta_\gamma = \Gamma_{S^1}(\gamma^*(\zeta)).$$

Perhaps the most important example of these bundles is when $\zeta = TM$ is the tangent bundle of M. The bundle LTM is the infinite dimensional tangent bundle of LM. The tangent space over γ is the space of vector fields living over γ.

Assume that M is simply connected, so that LM is connected. In this case, for any $\gamma \in LM$, the pull back, $\gamma^*(\zeta)$ is trivial. Choosing a trivialization gives us isomorphisms

$$L\zeta_\gamma = \Gamma_{S^1}(\gamma^*(\zeta)) \cong L\mathbb{C} \otimes \zeta_{\gamma(0)}. \tag{1.2}$$

Therefore, for each $\gamma \in LM$, we have a Fourier decomposition of the fiber, $L\zeta_{|_\gamma}$, induced by the subspace corresponding to $L_+\mathbb{C} \otimes \zeta_{\gamma(0)}$ with respect to such a trivialization. However, this will not necessarily vary continuously over LM.

The following corollary to the above theorem says that $L\zeta$ rarely will have a global Fourier decomposition.

COROLLARY 1.4. *If $L\zeta \to LM$ has a Fourier decomposition, then the map of based loop spaces,*

$$\Omega f_\zeta : \Omega M \to \Omega BU(n) \simeq U(n)$$

is null homotopic.

Remark. The homotopy type of the map of based loop spaces, $\Omega f_\zeta : \Omega M \to U(n)$ can be obtained by taking the holonomy map of a connection on ζ. Therefore this corollary can be interpreted as saying that in order for $L\zeta$ to have a Fourier decomposition, ζ must admit a "homotopy flat" connection.

The parallel transport operator induced by a connection on $\zeta \to M$ can be interpreted as an automorphism of the pull - back bundle $e^*(\zeta) \to LM$, where, as above, $e : LM \to M$ is the evaluation map. Let $\mathcal{G}(e^*(\zeta))$ be the gauge group of bundle automorphisms of $e^*(\zeta)$. Our final result shows how to deform the loop bundle $L\zeta$ by such an automorphism and examines its decomposability properties.

THEOREM 1.5. *Let $\zeta \to M$ be an n dimensional bundle over a smooth, simply connected manifold. Then there is a natural rank n loop bundle*

$$L^{\mathcal{G}}\zeta \to \mathcal{G}(e^*(\zeta)) \times LM$$

satisfying the following properties. For $\tau \in \mathcal{G}(e^*(\zeta))$, let $L^\tau M$ denote the restriction of $L^{\mathcal{G}}\zeta$ to $\{\tau\} \times LM$; then,

1. *For the identity element, $id \in \mathcal{G}(e^*(\zeta))$, $L^{id}\zeta = L\zeta \to LM$.*
2. *For τ_α the parallel transport operator of a connection α on ζ, the bundle $L^{\tau_\alpha}\zeta \to LM$ admits a natural isomorphism of loop bundles,*

$$L^\tau\zeta \cong L\mathbb{C} \otimes e^*\zeta$$

and hence it admits Fourier decomposition.

Remark. Given a connection on ζ with parallel transport operator τ, we refer to the resulting loop bundle $L^\tau\zeta$ as the "holonomy loop bundle" induced by this connection. Notice that the isomorphism type of these bundles is independent of the connection because the space of connections is connected. Hence all parallel transport operators coming from connections lie in the same path component of $\mathcal{G}(e^*(\zeta))$.

This paper is organized as follows. In section 2 we will give careful definitions of loop bundles and their Fourier decompositions and prove theorem 1.1 and its corollaries. Theorem 1.5 will be proved in section 3.

This paper was motivated by studying the beautiful ideas contained in J. Morava's work [**2**]. The main result of that paper was incorrect, and the author has since withdrawn it. However the ideas in that paper are deep and have influenced this present work greatly. The main goal of the project culminating in this paper was to clarify the situation regarding decompositions of the tangent bundle of the loop space of a manifold. The authors are grateful to Morava for many hours of discussion about loop bundles, holonomy, and related topics.

2. Fourier decompositions and a proof of theorem 1.1

In this section we define a rank n loop bundle and the concept of a Fourier decomposition on such a bundle. We conclude by proving theorem 1.1.

In what follows all of our spaces will be of the homotopy type of a CW - complex of finite type.

Definition 1. a. A *rank n loop bundle* $\xi \to X$ is a vector bundle over X with fiber isomorphic to $L\mathbb{C}^n$ and structure group $LU(n)$. The classifying space of such a bundle is the loop space $LBU(n)$.

b. Let $\xi \to X$ be a rank n loop bundle classified by a map $f_\xi : X \to LBU(n)$. Let $e : LBU(n) \to BU(n)$ be the map that evaluates a loop at $1 \in S^1$. The *underlying n-dimensional vector bundle*, $\xi_U \to X$, is the bundle classified by the composition $e \circ f_\xi : X \to LBU(n) \to BU(n)$.

In the language of principal bundles, a rank n loop bundle classified by a map $f_\xi : X \to LBU(n)$ is:

$$\xi = f_\xi^*(LEU(n)) \times_{LU(n)} L\mathbb{C}^n$$

where $EU(n) \to BU(n)$ is the universal $U(n)$-bundle. The underlying vector bundle is:

$$\xi_U = f_\xi^*(LEU(n)) \times_{LU(n)} \mathbb{C}^n = f_\xi^* e^*(EU(n)) \times_{U(n)} \mathbb{C}^n$$

where $LU(n)$ acts on $\mathbb{C}^n$ via the map $\gamma \cdot v = \gamma(1)v$.

As mentioned in the introduction, an important class of loop bundles arises in the following way: let $\zeta \to M$ be a finite dimensional vector bundle over a finite dimensional manifold M. Let $f_\zeta : M \to BU(n)$ be a classifying map for ζ. The loop space of ζ has the structure of a loop bundle over LM: $L\zeta \to LM$. It is classified by the map $Lf_\zeta : LM \to LBU(n) \simeq BLU(n)$. If $\zeta \to M$ is the tangent bundle of M, $L\zeta$ is the tangent bundle of LM.

Another interesting class of examples of loop bundles arises by taking a finite dimensional vector bundle $\zeta \to X$ and forming the fiberwise tensor product with $L\mathbb{C}$. In terms of classifying maps, this corresponds to the map $[X, BU(n)] \to$

$[X, LBU(n)]$ induced by the inclusion $BU(n) \to LBU(n)$ as the space of constant maps.

Since the structure group of our loop bundles is $LU(n)$, any such bundle has an inner product that on each fiber is isomorphic to the inner product on $L\mathbb{C}^n$ given by (1.1). This allows us to consider the existence of orthogonal complements of subbundles. Such complements may not necessarily exist, even when the subbundle is fiberwise closed for the pre-Hilbertian topology defined by the inner product. But when a complement does exist locally, it can be extended globally using the unitary structure.

Using the properties of the subspace $L_+\mathbb{C}^n \subset L\mathbb{C}^n$ described in the introduction as our fiberwise model, we now define the notion of a "*Fourier decomposition*" of a loop bundle.

Definition 2. A *Fourier decomposition* of a rank n loop bundle $\xi \to X$ is a subbundle $\psi \subset \xi$ satisfying the following properties:

(1) ψ has an orthogonal complement $\psi^\perp \subset \xi$ with $\xi = \psi \oplus \psi^\perp$.
(2) ψ is invariant under multiplication by $z \in \mathbb{C}[z, z^{-1}]$, $z\psi \subseteq \psi$. Furthermore $z\psi$ has codimension n in ψ.

Remarks. 1. The notation for the decomposition implies that ψ is fibrewise closed in the pre-Hilbertian topology.

2. We will see in our proof of theorem 1.1 that with the existence of such a subbundle, there exists a filtration

$$\cdots \subset z^{-k}\psi \subset z^{-(k+1)}\psi \subset \cdots \subset \xi$$

whose union $\bigcup_k z^{-k}\psi$ is a fiberwise dense subbundle of ξ. This is the bundle theoretic analogue of the Fourier decomposition of $L\mathbb{C}^n$.

3. A Fourier decomposition yields a *polarization* of ξ in the sense of [**3**]. However a Fourier decomposition is much stronger than a polarization since the splitting of a polarized bundle at a particular fiber need only be well defined up to a finite dimensional ambiguity. See [**3**] for details.

We are now ready to give a proof of theorem 1.1, as stated in the introduction. We begin with a lemma.

Lemma 2.1. *Let $W \oplus W^\perp$ be an orthogonal decomposition of $L\mathbb{C}^n$ with the property that $zW \subseteq W$ has codimension n. Then the space $W \cap zW^\perp$ is an n dimensional subspace of $L\mathbb{C}^n$. Furthermore, given a unitary isomorphism, $\phi : \mathbb{C}^n \to W \cap zW^\perp$, the following composition defines an element of $LU(n)$:*

$$L\mathbb{C} \otimes \mathbb{C}^n \xrightarrow[\cong]{1\otimes\phi} L\mathbb{C} \otimes (W \cap zW^\perp) \xrightarrow{\rho} L\mathbb{C}^n$$

Here ρ is the unique map of $L\mathbb{C}$-modules extending the inclusion $W \cap zW^\perp \hookrightarrow L\mathbb{C}^n$.

Proof. Consider the space $W \cap zW^\perp$. This lies inside W and is orthogonal to zW in W. As the map $L\mathbb{C}^n \to L\mathbb{C}^n$ defined by $\alpha \to z\alpha$ is a unitary isomorphism, $L\mathbb{C}^n$ can be orthogonally decomposed as $zW \oplus zW^\perp$. From this and the fact that $zW \subseteq W$ has codimension n, we deduce that there is an orthogonal decomposition $W = (W \cap zW^\perp) \oplus zW$. Hence $W \cap zW^\perp$ is n dimensional.

We continue with an argument adapted from [**3**, 8.3.2]. Let $\{w_1, \ldots, w_n\}$ be an orthogonal basis for $W \cap zW^\perp$. Let (w_{jl}) denote the sequence of Fourier coefficients for w_j. As each w_j is a smooth map $S^1 \to \mathbb{C}^n$, the sum $\sum_{l\in\mathbb{Z}} w_{jl}e^{il\theta}$ is absolutely

convergent and sums to $w_j(\theta)$. We consider the inner product in $\mathbb{C}^n$ of $w_j(\theta)$ and $w_k(\theta)$:

$$\begin{aligned}\langle w_j(\theta), w_k(\theta)\rangle &= \left\langle \sum_{l\in\mathbb{Z}} w_{jl}e^{il\theta}, \sum_{m\in\mathbb{Z}} w_{km}e^{im\theta} \right\rangle \\ &= \sum_{l,m\in\mathbb{Z}} \langle w_{jl}, w_{km}\rangle\, e^{i(m-l)\theta} \\ &= \sum_{p\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} \left\langle w_{jr}, w_{k(p+r)}\right\rangle e^{ip\theta} \\ &= \sum_{p\in\mathbb{Z}} (w_j, z^p w_k)\, e^{ip\theta} \\ &= \delta_{jk}\end{aligned}$$

the last identity uses the fact that the w_j are an orthonormal basis for $W \cap zW^\perp$ and that this space is orthogonal to zW within which the zw_j lie.

Hence $\{w_1(\theta), \ldots, w_n(\theta)\}$ is an orthonormal basis for $\mathbb{C}^n$ for each $\theta \in S^1$. Thus there is an element $\gamma \in LU(n)$ such that $\gamma(\theta)e_j = w_j(\theta)$. The fact that the w_j are smooth demonstrates that γ is smooth. The map defined in the statement of the proposition is easily seen to agree with γ. □

The proof of theorem 1.1 can now be completed quickly.

Proof of theorem 1.1. Let ξ have a Fourier decomposition $\psi \oplus \psi^\perp \cong \xi$. Consider the subspace, $\psi \cap z\psi^\perp \subset \xi$. We first show that $\psi \cap z\psi^\perp \to X$ is a subbundle of ξ. That is, we show that it is a locally trivial fibration with fiber a vector space of constant rank. Consider $V \subseteq X$ open over which the decomposition $\xi = \psi \oplus \psi^\perp$ can be trivialised; i.e. there is an isomorphism $\xi|_V \to V \times L\mathbb{C}^n$ which carries $\psi|_V$ to a subspace W and $\psi^\perp|_V$ to $W^\perp$. This map is $L\mathbb{C}$ equivariant and so $zW \subseteq W$ with codimension n. We note that $\psi \cap z\psi^\perp$ corresponds to $W \cap zW^\perp$ which is an n dimensional vector space. Thus $\psi \cap z\psi^\perp$ is a locally trivial fibration with fiber isomorphic to $\mathbb{C}^n$.

By lemma 6, the evaluation map, $e : L\mathbb{C}^n \to \mathbb{C}^n$, carries $W \cap zW^\perp$ isomorphically onto $\mathbb{C}^n$. This implies that the composition

$$\psi \cap z\psi^\perp \hookrightarrow \xi \xrightarrow{\ e\ } \xi_U$$

is an isomorphism.

Now notice that the map of $L\mathbb{C}$-module bundles $\rho : L\mathbb{C} \otimes (\psi \cap z\psi^\perp) \to \xi$ extending the inclusion $\psi \cap z\psi^\perp \hookrightarrow \xi$ corresponds on the fibers over V to:

$$L\mathbb{C}^n \to L\mathbb{C} \otimes \mathbb{C}^n \to L\mathbb{C} \otimes (W \cap zW^\perp) \xrightarrow{\ \rho\ } L\mathbb{C}^n$$

By lemma 2.1 this is an isomorphism. This implies

$$\rho : L\mathbb{C} \otimes (\psi \cap z\psi^\perp) \to \xi$$

is an isomorphism of loop bundles. Coupled with the isomorphism $e : \psi \cap z\psi^\perp \cong \xi_U$, we have an isomorphism of loop bundles

$$L\mathbb{C} \otimes \xi_U \cong \xi.$$

But, as observed earlier, the left hand bundle is classified by the composition

$$X \xrightarrow{\ f_\xi\ } LBU(n) \xrightarrow{\ e\ } BU(n) \xrightarrow{\ \iota\ } LBU(n)$$

where $\iota : BU(n) \hookrightarrow LBU(n)$ is the inclusion of the constant loops. This isomorphism implies that the structure group of ξ can be reduced to $U(n) \subset LU(n)$.

The converse of this statement is immediate. If the structure group of ξ can be reduced to $U(n) \subset LU(n)$ then $\xi \cong L\mathbb{C} \otimes \xi_U$. If $\psi \subset \xi$ is the subbundle corresponding to $L_+\mathbb{C} \otimes \xi_U$ then ψ defines a Fourier decomposition of ξ. □

Notice in the proof of this theorem we observed that if $\psi \subset \xi$ yields a Fourier decomposition then there is an isomorphism, $L\mathbb{C} \otimes (\psi \cap z\psi^{\perp}) \cong \xi$. This says that $\mathbb{C}[z, z^{-1}] \otimes (\psi \cap z\psi^{\perp})$ can be viewed as a dense subbundle of ξ or equivalently, there is a fiberwise dense filtration,

$$\cdots \subset z^{-k}\psi \subset z^{-(k+1)}\psi \subset \cdots \subset \xi.$$

This is the fiberwise analogue of the Fourier decomposition of $L\mathbb{C}^n$.

Remarks on the proofs of corollaries 1.2 - 1.4. As observed in the introduction, corollary 1.2 is simply a restatement of theorem 1.1. Note also that corollary 1.3 follows from our proof of theorem 1.1. To prove corollary 1.4, let $\Omega\zeta$ be the principal $\Omega U(n)$ bundle over the based loop space classified by $\Omega f_\zeta : \Omega M \to \Omega BU(n) \simeq U(n)$. The argument in the proof of theorem 1.1 proves that if $L\zeta$ has a Fourier decomposition determined by $\psi \subset L\zeta$, there is an isomorphism of vector bundles whose structure group is $\Omega U(n)$,

$$\rho : \Omega\mathbb{C} \otimes (\psi \cap z\psi^{\perp})_{|\Omega M} \cong \Omega\zeta. \tag{2.1}$$

But, as shown in the proof of theorem 1.1, the bundle $\psi \cap z\psi^{\perp}$ is classified by the composition

$$LM \xrightarrow{Lf_\zeta} LBU(n) \xrightarrow{e} BU(n)$$

which is homotopic to the composition

$$LM \xrightarrow{e} M \xrightarrow{f_\zeta} BU(n).$$

This composition is null homotopic when restricted to the based loop space, ΩM, so the restriction $(\psi \cap z\psi^{\perp})_{|\Omega M}$ is a trivial n-dimensional bundle. Thus (2.1) yields a trivialization of $\Omega\zeta$ as vector bundle with structure group $\Omega U(n)$. This implies the classifying map $\Omega f : \Omega M \to U(n)$ is null homotopic, which is the statement of corollary 1.4.

3. Twisting a loop bundle by a gauge transformation

Let $\zeta \to M$ be a smooth n - dimensional complex bundle classified by a map $f_\zeta : M \to BU(n)$. Let $L\zeta \to LM$ be the loop bundle described above. It is classified by the loop of f, $Lf : LM \to LBU(n)$. Let $e : LX \to X$ be the evaluation map and consider the pull -back bundle, $e^*(\zeta) \to LM$. In this section we describe how to twist the loop bundle $L\zeta$ by an automorphism of the bundle $e^*(\zeta)$ and we prove theorem 1.5 as stated in the introduction.

Let $\mathcal{G}(e^*(\zeta))$ be the gauge group of smooth bundle automorphisms of $e^*(\zeta)$. An element $g \in \mathcal{G}(e^*(\zeta))$ can be viewed as a diagram

$$\begin{array}{ccc} e^*(\zeta) & \xrightarrow{g} & e^*(\zeta) \\ \downarrow & & \downarrow \\ LM & \xrightarrow[=]{} & LM. \end{array}$$

We now go about defining the loop bundle $L^{\mathcal{G}}\zeta \to \mathcal{G}(e^*(\zeta)) \times LM$ discussed in theorem 1.5. We do this first by defining a classifying map

$$L^{\mathcal{G}} f_\zeta : \mathcal{G}(e^*(\zeta)) \times LM \to LBU(n).$$

Recall that $\mathcal{G}(e^*(\zeta))$ can be described as the group of sections of the adjoint bundle of the associated principal bundle to $e^*(\zeta)$. This principal bundle is the pull back of the universal bundle $EU(n) \to BU(n)$ via the composition $f_\zeta \circ e : LM \to BU(n)$. Let $Ad(\zeta)$ denote the corresponding fiber bundle:

$$Ad(\zeta) = ((f_\zeta \circ e)^*(EU(n)) \times_{Ad} U(n) \longrightarrow LM$$

where the notation $\times_{Ad}$ refers to taking the orbit space of the diagonal $U(n)$ action, where $U(n)$ acts freely on $(f_\zeta \circ e)^*(EU(n))$ and by conjugation on $U(n)$. The following is standard (see [**1**] for a good reference).

PROPOSITION 3.1. *There is a natural isomorphism of groups,*

$$\mathcal{G}(e^*(\zeta)) \cong \Gamma(Ad(\zeta)),$$

where the right hand side is the group of all smooth sections of the adjoint bundle. The multiplication in $\Gamma(Ad(\zeta))$ is the fiberwise product of sections.

Now for any group G, let $Ad(EG) = EG \times_{Ad} G$ be the corresponding adjoint bundle. The following is well known.

PROPOSITION 3.2. *There is a natural homotopy equivalence $h : Ad(EG) \to LBG$ making the following map of fibrations homotopy commute:*

$$\begin{array}{ccc} G & \xrightarrow[\simeq]{h} & \Omega BG \\ \downarrow & & \downarrow \\ Ad(EG) & \xrightarrow[\simeq]{h} & LBG \\ \downarrow & & \downarrow e \\ BG & \xrightarrow[=]{} & BG \end{array}$$

where $h : G \to \Omega BG$ is the usual homotopy equivalence, adjoint to the map $\Sigma G \to BG$ that classifies the G - bundle over ΣG with the single clutching function given by the identity $G \to G$.

By these two propositions we have the following composition:

$$\bar{\Phi}_\zeta : \mathcal{G}(e^*(\zeta)) \cong \Gamma(Ad(e^*\zeta)) \hookrightarrow Map(LM, Ad(e^*\zeta)) \xrightarrow{(f_\zeta \circ e)^*} Map(LM, Ad(EG)) \xrightarrow[\cong]{h} (LM, LBG). \tag{3.1}$$

Notice that the map $\bar{\Phi}_\zeta : \mathcal{G}(e^*(\zeta)) \to Map(LM, LBG)$ is well defined up to homotopy. Therefore the adjoint map

$$\Phi_\zeta : \mathcal{G}(e^*(\zeta)) \times LM \to LBG$$

is well defined up to homotopy and hence determines an isomorphism class of loop bundle

$$L^{\mathcal{G}}\zeta \to \mathcal{G}(e^*(\zeta)) \times LM. \tag{3.2}$$

We can describe the bundle $L^{\mathcal{G}}\zeta$ in a fiberwise manner as follows. Let $\gamma \in LM$. For $t \in S^1$, let $t\gamma$ be rotation of the loop γ by t. That is, $t\gamma(s) = \gamma(s+t)$. Now let $\tau \in \mathcal{G}(e^*(\zeta))$. Then $\tau(\gamma) : \zeta_{\gamma(0)} \to \zeta_{\gamma(0)}$ is an isomorphism. So for $t \in S^1$, $\tau(t\gamma) : \zeta_{\gamma(t)} = \zeta_{t\gamma(0)} \to \zeta_{t\gamma(0)} = \zeta_{\gamma(t)}$ is an isomorphism. Finally, let $\tilde{\gamma} : \mathbb{R} \to M$ be the composition $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \xrightarrow{\gamma} M$.

The fiber $L^{\mathcal{G}}\zeta_{|_{(\tau,\gamma)}}$ is given by

$$L^{\mathcal{G}}\zeta_{|_{(\tau,\gamma)}} = \{\sigma \in \Gamma(\tilde{\gamma}^*(\zeta)) : \text{ for each } t \in \mathbb{R},\ \sigma(t+1) = \tau(t\gamma)(\sigma(t))\}. \tag{3.3}$$

For $\tau \in \mathcal{G}(e^*(\zeta))$, let $L^\tau\zeta \to LM$ be the loop bundle given by the restriction of $L^{\mathcal{G}}\zeta$ to $\{\tau\} \times LM$. As can be seen by either considering the classifying map Φ_ζ or from the fiberwise description 3.3, one notices that for the identity $id \in \mathcal{G}(e^*(\zeta))$, $L^{id}(\zeta) = L\zeta \to LM$. Therefore we view the bundles $L^\tau\zeta$ as twistings or deformations of the loop bundle $L\zeta$ by the automorphism τ.

Interpretation. Each $\tau \in \mathcal{G}(e^*(\zeta))$ is an automorphism of $e^*\zeta$ and therefore determines a fiberwise action of the integers $\mathbb{Z}$ on $e^*\zeta \to LM$. That is, for each loop γ there is a representation of $\mathbb{Z}$ on $\zeta_{\gamma(0)}$ given by $\tau(\gamma)$. In particular this induces a fiberwise action of $\mathbb{Z}$ on $\tilde{\gamma}^*\zeta \to \mathbb{R}$, where the representation of the fiber over $t \in \mathbb{R}$, which is $\zeta_{\gamma(t)}$, is given by $\tau(t\gamma)$. Of course there is the free action of $\mathbb{Z}$ on $\mathbb{R}$, and the fiber at each loop, $L^\tau\zeta_\gamma$, consists of the $\mathbb{Z}$-equivariant sections of $\tilde{\gamma}^*(\zeta)$:

$$(L^\tau\zeta)_\gamma = \Gamma_{\mathbb{Z}}(\tilde{\gamma}^*\zeta).$$

When $\tau = id$, the resulting $\mathbb{Z}$-action on each $\tilde{\gamma} * \zeta$ is trivial and so the fiber of the *untwisted* loop bundle, $L\zeta$, at $\gamma \in LM$ consists of the *invariant* sections of $\tilde{\gamma}^*(\zeta)$, which are the sections of $\gamma^*(\zeta) \to \mathbb{R}/\mathbb{Z}$.

An important example of a deformation (twisting) of the loop bundle comes from a connection α on ζ. For such a connection, let τ_α be the parallel transport operator. We can view τ_α as an element of the gauge group $\mathcal{G}(e^*(\zeta))$, by defining for every $\gamma \in LM$, $\tau_\alpha(\gamma) : \zeta_{\gamma(0)} \to \zeta_{\gamma(0)}$ to be the holonomy operator around the loop γ determined by the connection α. We refer to the corresponding loop bundle $L^{\tau_\alpha}\zeta$ as the *holonomy loop bundle* associated to the connection α. Notice that since the space of connections is affine, and therefore connected, the subspace of $\mathcal{G}(e^*(\zeta))$ determined by connections on ζ is connected. Therefore the isomorphism class of the holonomy bundle $L^{\tau_\alpha}\zeta$ is independent of the choice of connection α.

We now discuss the equivariance of the bundle $L^{\mathcal{G}}\zeta$. Consider the usual rotation action on the loop space, $S^1 \times LM \to LM$, and the trivial S^1 action on $\mathcal{G}(e^*(\zeta))$. We then take the diagonal action of S^1 on the product, $\mathcal{G}(e^*(\zeta)) \times LM$.

Proposition 3.3. *The loop bundle $L^{\mathcal{G}}\zeta$ has an $\mathbb{R}$-action, yielding an action map of loop bundles,*

$$\begin{array}{ccc} \mathbb{R} \times_{\mathbb{Z}} L^{\mathcal{G}}\zeta & \longrightarrow & L^{\mathcal{G}}\zeta \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} \times (\mathcal{G}(e^*(\zeta)) \times LM) & \longrightarrow & \mathcal{G}(e^*(\zeta)) \times LM. \end{array}$$

Proof. To define the action we consider the fiberwise description of $L^{\mathcal{G}}\zeta$ given above (3.3). Let $\sigma \in L^{\mathcal{G}}\zeta_{(\tau,\gamma)}$, where $(\tau,\gamma) \in \mathcal{G}(e^*(\zeta)) \times LM$. Let $t \in \mathbb{R}$. Define $t\sigma \in L^{\mathcal{G}}\zeta_{(\tau,t\gamma)}$ by $t\sigma(s) = \sigma(s+t)$ for $s \in \mathbb{R}$. It is immediate that this defines an action that satisfies the required properties. □

Notice that since the circle action on $\mathcal{G}(e^*(\zeta))$ is trivial, this induces an $\mathbb{R}$-action on each twisted loop bundle, $L^\tau\zeta$ for $\tau \in \mathcal{G}(e^*(\zeta))$. In particular for $\tau = id$, this action descends to the usual circle action $\mathbb{R}/\mathbb{Z} \times L\zeta \to L\zeta$. For $\zeta = TM$, then $LTM = TLM$ is the tangent bundle of the loop space and this action is the tangential action induced by rotation of loops. The twisted tangent loop bundles, $L^\tau TM$, do not in general have S^1-actions, but only $\mathbb{R}$-actions.

Notice that besides the $\mathbb{R}$-action, each twisted bundle, $L^\tau\zeta$, is a module bundle over the ring of functions, $L\mathbb{C}$. The action is given by pointwise multiplication of sections. That is, if $f \in L\mathbb{C}$, and $\sigma \in \{\Gamma(\tilde{\gamma}^*(\zeta)) : \text{ for each } t \in \mathbb{R},\ \sigma(t+1) = \tau(t\gamma)(\sigma(t))\} = L^\tau\zeta_{|_\gamma}$, then

$$f \cdot \sigma(t) = f(t)\sigma(t) \tag{3.4}$$

is a section of $\tilde{\gamma}^*(\zeta)$ living in $L^\tau\zeta_{|_\gamma}$.

The following result will complete the proof of theorem 1.5, as stated in the introduction.

Proposition 3.4. *Let $\tau_\alpha \in \mathcal{G}(e^*(\zeta))$ be the parallel transport operator of a connection α on ζ. Then there is an $\mathbb{R}$-equivariant isomorphism of $L\mathbb{C}$-module loop bundles,*

$$j : e^*\zeta \otimes L\mathbb{C} \to L^{\tau_\alpha}\zeta.$$

Proof. We begin by defining the embedding $j : e^*\zeta \hookrightarrow L^{\tau_\alpha}\zeta$. For $\gamma \in LM$ and $t \in \mathbb{R}$, let $\tau_\alpha^{[0,t]}(\gamma) : \zeta_{\gamma(0)} \to \zeta_{\gamma(t)}$ denote the parallel transport operator along the curve γ between 0 and t. We then define $j : e^*\zeta \hookrightarrow L^{\tau_\alpha}\zeta$ in a fiberwise way as follows. Let $v \in \zeta_{\gamma(0)} = e^*\zeta_\gamma$. Define $j(v) \in \Gamma(\tilde{\gamma}^*(\zeta))$ by

$$j(v)(t) = \tau_\alpha^{[0,t]}(\gamma)(v) \in \zeta_{\gamma(t)} = \tilde{\gamma}^*(\zeta)_t.$$

By construction, $j(v)(t+1) = \tau_\alpha(t\gamma)(j(v)(t))$ and so $j(v) \in L^{\tau_\alpha}(\zeta)_\gamma$. Clearly this defines an embedding of bundles over LM, $j : e^*\zeta \hookrightarrow L^{\tau_\alpha}\zeta$. Moreover, this embedding is $\mathbb{R}$-equivariant where $\mathbb{R}$ acts on $e^*(\zeta)$ by $t \cdot v = \tau_\alpha^{[0,t]}(\gamma)(v)$ for $v \in (e^*\zeta)_\gamma = \zeta_{\gamma(0)}$. Using the $L\mathbb{C}$-module structure, we now extend this embedding to:

$$j : e^*\zeta \otimes L\mathbb{C} \to L^{\tau_\alpha}\zeta.$$

This is an $\mathbb{R}$-equivariant map of $L\mathbb{C}$-module loop bundles. We now only need to verify that it is an isomorphism. For this we define an inverse map on a fiber over a loop $\gamma \in LM$. The fiber over the left hand side, $e^*\zeta \otimes L\mathbb{C}$ at γ, is $\zeta_{\gamma(0)} \otimes L\mathbb{C} = L\zeta_{\gamma(0)}$. The fiber over the right hand side, $(L^{\tau_\alpha}\zeta)_\gamma$, is given by the space of sections $\{\sigma \in \Gamma(\tilde{\gamma}^*(\zeta)) : \text{ for each } t \in \mathbb{R},\ \sigma(t+1) = \tau_\alpha(t\gamma)(\sigma(t))\}$ (see (3.3)). We now define an inverse map

$$\Phi_\gamma : (L^{\tau_\alpha}\zeta)_\gamma \longrightarrow L\zeta_{\gamma(0)}. \tag{3.5}$$

For $t \in \mathbb{R}$, let $-\tilde{\gamma}_t : [0,t] \to M$ be given by $-\tilde{\gamma}_t(s) = \tilde{\gamma}(t-s)$. Then define

$$\Phi_\gamma(\sigma)(t) = \tau_\alpha^{[0,t]}(-\tilde{\gamma}_t)(\sigma(t)) \in \zeta_{\gamma(0)},$$

where, as before, $\tau_\alpha^{[0,t]}(-\tilde{\gamma}_t) : \zeta_{\gamma(t)} \to \zeta_{\gamma(0)}$ is parallel transport along the path $-\tilde{\gamma}_t$ via the connection α. To check it is well defined, we need to see that $\Phi_\gamma(\sigma)(t+1) = \Phi_\gamma(\sigma)(t)$ for all $t \in \mathbb{R}$. This is true because $\sigma(t+1) = \tau_\alpha(t\gamma)(\sigma(t))$. One now immediately sees that Φ_γ is inverse to $j_\gamma : \zeta_{\gamma(0)} \otimes L\mathbb{C} \to (L^{\tau_\alpha}\zeta)_\gamma$. □

Remark. Notice that this theorem says that the holonomy loop bundle $L^{\tau_\alpha}\zeta$ has a Fourier decomposition for any connection α. On the other hand, corollary 1.3 of theorem 1.1 says that the *untwisted* loop bundle $L\zeta = L^{id}\zeta$ has a Fourier decomposition if and only if $L\zeta \cong \zeta \otimes L\mathbb{C} \cong L^{\tau_\alpha}\zeta$. Viewing the holonomy loop bundle $L^{\tau_\alpha}\zeta$ as a *deformation* of $L\zeta$ by means of the bundle $L^{\mathcal{G}}\zeta$ over $\mathcal{G} \times LM$, we see that $L\zeta$ has a Fourier decomposition if the parallel transport operator τ_α lies in component of the identity in $\mathcal{G}$.

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