# Finite Dimensional Subbundles of Loop Bundles

Andrew Stacey

October 13, 2003

**Abstract**

In this paper we consider vector bundles with fibre a loop space and structure group a loop group. We determine necessary and sufficient conditions for the structure group to reduce to the group of constant loops in terms of the existence of certain finite dimensional subbundles of the original vector bundle.
Primary MSC Classification (2000): 58D15, 58B25

## 1 Introduction

In this paper we study subbundles of loop bundles. Following [CS02], we define a rank $n$ loop bundle to be a vector bundle $\xi \to X$ with fibre $L\mathbb{C}^n$ and structure group $LGl_n$. We will find it convenient to assume a reduction of the structure group to $LU_n$ which is equivalent to choosing a fibrewise inner product on $\xi$ isomorphic in charts to the standard inner product on $L\mathbb{C}^n$.

The problem we wish to consider is this: suppose that $\zeta \to X$ is a finite dimensional vector bundle over $X$; what are the implications for the structure groups of $\zeta$ and of $\xi$ of the statement that $\zeta$ is a subbundle of $\xi$?

The idea behind this problem was to study a loop bundle $\xi$ by considering an increasing sequence of finite dimensional subbundles $\xi_k$ which approximate it. The aim is to use this sequence to transfer finite dimensional constructions to the infinite via a limiting sequence, much in the spirit of the construction of the Witten genus in [Wit88]. This idea of approximating an infinite dimensional vector bundle by a filtration of finite dimensional subbundles has proved very successful in examining the structure of Hilbert manifolds, see for example [Elw68] and [EE70]. The vector bundles we consider inject naturally into bundles of Hilbert spaces and thus can be trivialised using the contractibility of the unitary group of a Hilbert space, [Kui65]. One can then impose a filtration, referred to as a layer structure in [EE70], and ask whether this filtration is in fact one of the original loop bundle. Under certain reasonable conditions on the filtration, we show that it can only be so in very special circumstances.

Our reasonable condition imposed on the $\xi_k$ is that there is some $k$ such that the family $\{z^n\xi_k : n \in \mathbb{Z}\}$ is fibrewise dense in $\xi$. Our main result can be summarised thus: if there is a finite dimensional subbundle $\zeta$ for which

$\{z^n\zeta : n \in \mathbb{Z}\}$ is fibrewise dense in $\xi$ then the structure group of $\xi$ reduces from $LU_n$ to $U_n$.

The main example of a loop bundle is the tangent bundle of the free loop space of a smooth finite dimensional manifold $M$. In [Mor01] and [CS02], the question arose as to the existence of a circle equivariant subbundle of $TLM$ non-equivariantly isomorphic to the pull back of $TM$ via an evaluation map. In [CS02], the existence of such a bundle was found to be a strong condition, one of the implications being that the tangent bundle of the based loop space is parallelisable. In this paper we consider a more general setting but the conclusion stays the same.

In order to state our result precisely, we need to define a class of subspaces of $L\mathbb{C}^n$:

**Definition 1.1.** A finite dimensional subspace $W$ of $L\mathbb{C}^n$ is *almost generating* if there is some dense subset $T$ of $S^1$ such that the natural map $C^\infty(T, \mathbb{C}) \otimes W \to C^\infty(T, \mathbb{C}^n)$ is surjective. It is *generating* if we can take $T = S^1$.

Here we are using the natural restriction map from $L\mathbb{C}^n = C^\infty(S^1, \mathbb{C}^n)$ to $C^\infty(T, \mathbb{C}^n)$ in order to consider $W$ as a subspace of $C^\infty(T, \mathbb{C}^n)$. We will see that we can take $T$ open in $S^1$. This concept can be extended fibrewise to finite dimensional subbundles of loop bundles leading to:

**Theorem 1.2.**
1. *A loop bundle $\xi \to X$ admits an almost generating subbundle $\zeta \to X$ if and only if the structure group of $\xi$ reduces to $U_n$, viewed as the constant loops in $LU_n$. In this case, there is an isomorphism $\xi \cong L\mathbb{C} \otimes \psi$ where $\psi$ is an $n$-dimensional vector bundle over $X$.*
2. *A loop bundle $\xi \to X$ admits a reduction of its structure group to a compact group if and only if it admits a reduction of its structure group to $U_n$.*
3. *If $\zeta$ is a finite dimensional subbundle of a loop bundle $\xi$ then the structure group of $\zeta$ reduces to $U_{n_1} \times \cdots \times U_{n_k}$ for $n_1, \ldots, n_k \leq n$ with $n_1 + \cdots + n_k = \dim \zeta$.*

We note that we are being very precise with the notion of subbundles. We shall expand on this in section 2.

Our method of attack for this problem is to reduce it to the question of homomorphisms $\rho : G \to LU_n$ where $G$ is a compact Lie group. The adjoint map is thus a path of homomorphisms $\tilde{\rho} : S^1 \times G \to U_n$. From this point of view, the problem bears a strong resemblance to the study of subgroups of compact Lie groups using homology theory as found in [Dyn52], [Bro78], and [Gel91]. Our question is closest to that studied by [Dyn52] although we consider a slightly different angle and in fact use [Gel91] at a key step.

In this paper we work over the field of complex numbers. This is mostly for linguistic reasons as it allows us to avoid the phrases like "unitary (resp. orthogonal)" which would otherwise pervade this text. We can always complexify a real representation to apply the theorems above to the real case. Thus we

shall henceforth assume $\mathbb{C}$ and write, for example, $M_n$ for $M_n(\mathbb{C})$, the algebra of complex, $n \times n$ matrices.

Again for linguistic and notational purposes it is convenient to fix a category of differentiability of maps. All results hold in the continuous category and any stronger but as the question was motivated by considering the smooth category, we shall remain with smooth. Thus for a smooth manifold $M$, $LM$ denotes the space of smooth loops in $M$. We shall often use the natural inclusions $U_n \subseteq LU_n \subseteq LGl_n \subseteq LM_n$ and we shall identify each space with its image in $LM_n$ unless otherwise stated.

The paper is organised as follows: in section 2 we consider the definition of subbundles and show how this translates our question into one on Lie groups. We also prove theorem 1.2, part 3. In section 3 we prove theorem 1.2, parts 1 and 2.

# 2 Subbundles

The aspect of vector bundles and their subbundles that we wish to emphasise is the rôle of the structure group. For finite dimensional vector bundles this is an unnecessary elaboration because the structure group is usually assumed to be $Gl_n$. In infinite dimensions, the general linear group $Gl(E)$ is in general either contractible or not a regular Lie group and therefore one usually has a smaller Lie group as structure group. In this case, not all candidates for subbundles will be compatible with the smaller structure group. We make this precise by considering the standard definitions. For simplicity, we assume that $X$ is a connected smooth manifold.

**Definition 2.1.** 1. A vector bundle over $X$ with typical fibre $V$ and structure group $G$ consists of a smooth manifold $\xi$, a surjective map $\pi : \xi \to X$, and a smooth atlas $\mathcal{U}$ of $X$ such that:

   (a) for $U \in \mathcal{U}$ there is a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times V$ such that $\pi|_{\pi^{-1}(U)} = p_U \circ \phi_U$,

   (b) for $U_1, U_2 \in \mathcal{U}$ with $U_1 \cap U_2 \neq \emptyset$, the adjoint of the map $\phi_1 \circ \phi_2^{-1} : U_1 \cap U_2 \times V \to U_1 \cap U_2 \times V$ is a map $U_1 \cap U_2 \to G$.

2. A vector bundle $\xi \to X$ with structure group $G$ admits a reduction to $H \subseteq G$ if there is an atlas $\mathcal{V}$ of $X$ compatible with $\mathcal{U}$ - that is, take a maximal atlas $\mathcal{U}'$ containing $\mathcal{U}$ for which item 1 holds and require $\mathcal{V} \subseteq \mathcal{U}'$ - such that item 1b holds with $\mathcal{V}$ replacing $\mathcal{U}$ and $H$ replacing $G$.

3. To say that $\zeta$ is a subbundle of $\xi$ is to give a smooth atlas $\mathcal{V}$ of $X$ compatible with the atlases $\mathcal{U}_\zeta$ of $\zeta$ and $\mathcal{U}_\xi$ of $\xi$, an inclusion map $i : \zeta \to \xi$ such that $\pi_\xi \circ i = \pi_\zeta$, and a linear inclusion $j : V_\zeta \to V_\xi$ such that for $U \in \mathcal{V}$, $(\mathrm{id} \times j) \circ \phi_U^\zeta = \phi_U^\xi \circ i : \pi_\zeta^{-1}(U) \to U \times V_\xi$.

The key ingredient in all of the above is the compatibility betwixt the atlases. From these definitions, the following proposition is immediate:

**Proposition 2.2.** *Suppose that $(\zeta, V_\zeta, G_\zeta)$ is a subbundle of $(\xi, V_\xi, G_\xi)$ over $X$. Then $\xi$ admits a reduction to the stabiliser group of $V_\zeta \subseteq V_\xi$ in $G_\xi$ and $\zeta$ admits a reduction to the image of this group in $G_\zeta$.*

As a consequence of this proposition, to prove theorem 1.2, part 1, it is sufficient to show that the stabiliser subgroup of an almost generating subspace of $L\mathbb{C}^n$ is conjugate in $LU_n$ to a subgroup of $U_n$. To prove part 2, it is then sufficient to show that any compact subgroup of $LU_n$ has an invariant, generating subspace. This we do in section 3. To prove theorem 1.2 part 3, we need to examine the image of the stabiliser group of a general $V \subseteq L\mathbb{C}^n$ in $Gl(V)$. This is simpler and we do that here.

*Proof of theorem 1.2 part 3.* Let $V \subseteq L\mathbb{C}^n$ be a finite dimensional subspace and $G \subseteq LU_n$ its stabiliser subgroup. For any $t \in S^1$, the evaluation map $e_t : L\mathbb{C}^n \to \mathbb{C}^n$ is $LU_n$-equivariant and thus $G$-equivariant. Using this, we can decompose $V$ into orthogonal subspaces $V_1 \oplus \cdots \oplus V_k$ with $\dim V_k \leq n$ and where $G$ acts on $V_i$ via an evaluation map $e_{t_i} : G \to U_n$. To do this, we fix some $t_1 \in S^1$ for which $e_{t_1} : V \to \mathbb{C}^n$ is not the zero map. The kernel of this is a $G$-equivariant subspace of $V$. As $G$ acts unitarily on $V$, the orthogonal complement of $V$ is also $G$-invariant and is carried $G$-equivariantly into $\mathbb{C}^n$ by $e_{t_1}$. As $V$ is finite dimensional, induction on the dimension of $V$ yields the above decomposition. The image of $G$ thus lies in $U_{n_1} \times \cdots U_{n_k}$ lying inside $U(V)$ in the natural way. The integers $n_i$ satisfy $n_i \leq n$ for all $i$ and $n_1 + \cdots + n_k = \dim V$.

This is the best general description as can be shown by the following case: let $k, m, n_1, \ldots, n_k \in \mathbb{N}$ be such that $n_1 + \cdots + n_k = m$. Choose $k$ distinct, non-void, closed, connected subsets $T_1, \ldots, T_k$ of $S^1$ and subordinate bump functions $\rho_1, \ldots, \rho_k : S^1 \to [0, 1]$. Let $V_i = \langle \rho_i e_1, \ldots, \rho_i e_{n_i} \rangle$ and $V = V_1 \oplus \cdots \oplus V_k$ (note that as the $T_i$ are distinct, the sum is direct). As $U_{n_i}$ is connected and the $T_i$ are distinct, for $x \in U_{n_i}$ there is some $g : S^1 \to U_n$ such that $g|_{T_j} = 1$ for $j \neq i$ and $g|_{T_i} = x$. By construction, $g \in G$ and thus the image of $G$ in $U(V)$ is precisely $U_{n_1} \times \cdots \times U_{n_k}$. □

# 3 Compact Lie Groups

In this section we prove theorem 1.2, parts 1 and 2. Our argument proceeds as follows: in section 3.1 we show that in $LM_n$, finite dimensional subalgebras have a very precise description. This gives a similar description for the group of units of such an algebra. In section 3.2, we show that the stabiliser algebra in $LM_n$ of a finite dimensional subspace of $L\mathbb{C}^n$ is finite dimensional if and only if the subspace is almost generating. This leads to the proof of theorem 1.2, part 1. In section 3.3, we adapt the Peter-Weyl theorem to show that any compact subgroup of $LU_n$ has an almost generating invariant subspace. This leads to the proof of theorem 1.2 part 2. We also give an interpretation of this in terms of compact subgroups of $LU_n$ and $\Omega U_n$, namely:

**Theorem 3.1.** *1. The only compact subgroups of $LU_n$ are conjugate to subgroups of $U_n$, and*

*2. there are no compact subgroups of $\Omega U_n$.*

We note that although we work with $LU_n$ throughout, parts of our results work for $LGl_n$ as well. We work with $LU_n$ throughout because from the topological point of view there is no difference between $LU_n$ and $LGl_n$ and the theory for $LU_n$ is slightly simpler than that for $LGl_n$.

## 3.1 Finite Dimensional Subalgebras of $LM_n$

The aim of this section is to prove the following theorem:

**Theorem 3.2.** *Let $\rho : G \to LU_n$ be a Lie group homomorphism such that the image of $\rho$ lies in a finite dimensional subalgebra of $LM_n$. Then there is some $\gamma \in LU_n$ and a Lie group homomorphism $\tilde{\rho} : G \to U_n$ such that $\rho = c_\gamma \tilde{\rho}$, where $c_\gamma : LU_n \to LU_n$ is the automorphism corresponding to conjugation by $\gamma$.*

**Corollary 3.3.** *Let $\xi \to X$ be a vector bundle with fibre $L\mathbb{C}^n$ and structure group $LU_n$. If $\xi$ admits a reduction of structure group to $G$ where $G \subseteq LU_n$ is contained in a finite dimensional subalgebra of $LM_n$ then $\xi$ admits a reduction of structure group to $U_n$ and there is an isomorphism of vector bundles $\xi \cong L\mathbb{C} \otimes \psi$ where $\psi \to X$ is an $n$-dimensional vector bundle.*

*Proof.* From theorem 3.2, $G$ is a subgroup of $U_n$ and is included in $LU_n$ via a conjugation of the natural one. If $P \to X$ is the principal $G$-bundle then the representations defined by the natural inclusion in $LU_n$ and the conjugated inclusion are isomorphic. Thus $\xi \cong P \times_G L\mathbb{C}^n$ where $G$ acts on $L\mathbb{C}^n$ by constant loops. Within this lies the the bundle $\psi \to X$ defined by $P \times_G \mathbb{C}^n$ and moreover $P \times_G L\mathbb{C}^n \cong L\mathbb{C} \otimes (P \times_G \mathbb{C}^n)$. Hence $\xi \cong L\mathbb{C} \otimes \psi$ as required. □

In [CS02], the bundle $\psi$ is referred to as the *underlying (finite dimensional) vector bundle* associated to $\xi$.

The key to the proof of theorem 3.2 is to consider the characteristic polynomial of $a(t)$ for $a \in LU_n$. Define the map $p : M_n \to P_n$ from $n \times n$ matrices to monic polynomials of degree $n$ to be the map which sends a matrix to its characteristic polynomial. This map is smooth and thus defines a smooth map $Lp : LM_n \to LP_n$.

**Lemma 3.4.** *Let $a \in LM_n$ have a minimum polynomial. Then the characteristic polynomial of $a(t)$ does not depend on $t$.*

*Proof.* Let $p(x)$ be the minimum polynomial of $a$, so $p(a) = 0$. Evaluation at $t \in S^1$ yields $p(a(t)) = 0$ and so the minimum polynomial of $a(t)$ divides $p(x)$. Thus the characteristic polynomial of $a(t)$ is a degree $n$ monic divisor of $p(x)^n$. As such divisors are classified by their roots, the set is discrete. Hence $Lp(a)$ is a constant loop in $P_n$. □

**Corollary 3.5.** *For such an $a$, $a \in LGl_n$ if and only if $a(t_0) \in Gl_n$ for some $t_0 \in S^1$.*

*Proof.* A matrix is invertible if and only if 0 is not a root of its characteristic polynomial. As the characteristic polynomial of $a(t)$ is the same as that of $a(t_0)$, $a(t)$ is invertible for all $t$ if and only if $a(t_0)$ is invertible. □

**Lemma 3.6.** *Let $a \in LU_n$ have a minimum polynomial. Then $a(t)$ is conjugate to $a(0)$ for all $t$.*

*Proof.* As $a \in LU_n$, $a(t) \in U_n$ for all $t$. As unitary operators are diagonalisable, each $a(t)$ is conjugate to a diagonal matrix, with diagonal entries given by the roots of the characteristic polynomial of $a(t)$. As this is independent of $t$, each $a(t)$ is conjugate to the same diagonal matrix and hence to $a(0)$. □

**Corollary 3.7.** *Let $a \in LU_n$ have a minimum polynomial. If $a(t_0) = 1$ for some $t_0 \in S^1$ then $a(t) = 1$ for all $t \in S^1$.*

*Proof of theorem 3.2.* Assume without loss of generality that $\rho : G \to LU_n$ is injective. Let $g \in \rho(G)$ considered as an element of $LM_n$. As this lies in a finite dimensional subalgebra of $LM_n$, the subalgebra generated by $g$ is finite dimensional and hence $g$ has a minimum polynomial in $LM_n$.

By corollary 3.7, $e_t\rho : G \to U_n$ is an injective Lie group homomorphism for all $t \in S^1$. Moreover, by lemma 3.6, the homomorphisms $e_t\rho$ are what is called *pointwise conjugate*; that is, for each $g \in G$ and $t \in S^1$ there is some $x_{g,t} \in U_n$ such that $e_t\rho(g) = x_{g,t}^{-1} e_0\rho(g) x_{g,t}$.

From [Gel91], there is thus a path $\gamma : [0,1] \to U_n$ with $\gamma(0) = 1$ such that $c_{\gamma(t)} e_0\rho = e_t\rho$, where $c_{\gamma(t)} : U_n \to U_n$ is conjugation by $\gamma(t)$. Since $e_1 = e_0 : LU_n \to U_n$, $e_1\rho = e_0\rho$ and thus $\gamma(1)$ lies in the centraliser of $H = e_1\rho(G)$ in $U_n$.

Let $g \in U_n$ lie in the centraliser of $H$. Thus $g$ commutes with all elements of $H$. Hence the eigenspaces of $g$ are $H$-invariant subspaces of $\mathbb{C}^n$ and so any element of $U_n$ which acts by scalars on the eigenspaces of $g$ will also commute with all elements of $H$ and so lie in the centraliser of $H$. There is a path from the identity to $g$ through such elements and thus the centraliser of $H$ in $U_n$ is connected.

Thus we can find a path $\beta$ from $1 = \gamma(0)$ to $\gamma(1)$ lying in the centraliser of $H$. Thus $c_{\gamma(t)\beta(t)} e_0\rho = c_{\gamma(t)} e_0\rho$ and so replacing $\gamma$ by $\gamma' = \gamma\beta$ yields a loop in $U_n$ such that $c_{\gamma'(t)} e_0\rho = e_t\rho$.

Thus $\rho = c_{\gamma'(t)} e_0\rho = c_{\gamma'(t)}\tilde{\rho}$, where $\tilde{\rho} = e_0\rho : G \to U_n$ is a Lie group homomorphism. □

## 3.2 Almost Generating Subspaces of $L\mathbb{C}^n$

In this section, we consider finite dimensional almost generating subspaces of $L\mathbb{C}^n$. Firstly, we give an alternative way to characterise them and note a subclass of $S^1$-equivariant generating subspaces. We use this characterisation to prove the following proposition from which we deduce theorem 1.2, part 1.

**Proposition 3.8.** *Let $V \subseteq L\mathbb{C}^n$ be a finite dimensional subspace. The stabiliser algebra of $V$ is finite dimensional if and only if $V$ is almost generating.*

Combined with theorem 3.2, this yields theorem 1.2, part 1, because if $\rho : G \to LU_n$ is such that there is a $G$-invariant almost generating subspace of $L\mathbb{C}^n$ then $\rho(G)$ lies in the stabiliser subalgebra of that subspace. From theorem 3.2, this implies that $\rho$ is conjugate in $LU_n$ to a map which factors through the inclusion $U_n \to LU_n$.

**Proposition 3.9.** *A finite dimensional subspace $V$ of $L\mathbb{C}^n$ is almost generating if and only if there is a dense subset $T$ of $S^1$ such that the maps $e_t|_V \colon V \to \mathbb{C}^n$ are surjective for $t \in T$.*

As part of the proof, we will see that the set $T$ can be taken to be open.

*Proof.* Let $V$ be a finite dimensional subspace of $L\mathbb{C}^n$ of dimension $m$. Consider the function $r : S^1 \to \mathbb{N}$ given by $r : t \to \dim \operatorname{Im} e_t|_V$. Choose a basis $\{v_i : 1 \leq i \leq m\}$ for $V$.

Let $t \in S^1$. As $\{v_i(t) : 1 \leq i \leq m\}$ spans $e_t(V)$ which is of dimension $r(t)$, there is some subset $N_t$ of $\{1, \ldots, m\}$ of size $r(t)$ such that $\{v_i(t) : i \in N_t\}$ is a basis for $e_t(V)$. In particular, $\{v_i(t) : i \in N_t\}$ is a linearly independent set. As the $v_i$ are continuous, there is some neighbourhood $U_t$ of $t$ for which $\{v_i(s) : i \in N_t\}$ is linearly independent for $s \in U_t$.

One direct consequence of this is that for $s \in U_t$, $\dim e_s(V) \geq r(t)$. Hence the function $r$ has a local minimum at $t$. As $t$ was arbitrary, $r$ has local minima at all points of $S^1$. In particular, the set $T = r^{-1}(n)$ is open in $S^1$.

Using the method prescribed, there is an open covering $\mathcal{U}$ of $T$ such that for each $U \in \mathcal{U}$, there is an $n$ element subset $N_U$ of $\{1, \ldots, m\}$ with the property that for $t \in U$, $\{v_i(t) : i \in N_I\}$ is a basis for $\mathbb{C}^n$. Choose a smooth partition of unity subordinate to $\mathcal{U}$, $\{\rho_\lambda : \lambda \in \Lambda\}$. For $\lambda \in \Lambda$, let $U_\lambda \in \mathcal{U}$ be such that the support of $\rho_\lambda$ is contained in $U_\lambda$. Let $N_\lambda$ be the corresponding $n$ element subset of $\{1, \ldots, m\}$.

Let $w \in C^\infty(T, \mathbb{C}^n)$. For $\lambda \in \Lambda$, as $\{v_i(t) : i \in N_\lambda\}$ is a basis for $\mathbb{C}^n$ for $t \in U_\lambda$, there are smooth functions $f_i^\lambda : U_\lambda \to \mathbb{C}$ such that $\sum_{i \in N_\lambda} f_i^\lambda(t) v_i(t) = w(t)$ for $t \in U_\lambda$. For $i \notin N_\lambda$, define $f_i^\lambda = 0$ on $U_\lambda$. Let $f_i : T \to \mathbb{C}$ be the smooth function $f_i = \sum_{\lambda \in \Lambda} \rho_\lambda f_i^\lambda$. As the $\rho_\lambda$ are a partition of unity, $\sum_{i=1}^m f_i(t) v_i(t) = w(t)$ for $t \in T$ and $w$ lies in the image of the map $C^\infty(T, \mathbb{C}) \otimes V \to C^\infty(T, \mathbb{C}^n)$ which is thus demonstrated to be surjective.

Conversely, suppose that $C^\infty(T, \mathbb{C}) \otimes V \to C^\infty(T, \mathbb{C}^n)$ is surjective for some $T \subseteq S^1$. Choose $w \in \mathbb{C}^n$ and $t \in T$. Denote by $w_c$ the constant map $T \to \mathbb{C}^n$, $t \to w$. From the surjectivity and using the basis $\{v_1, \ldots, v_m\}$ of V, there are $f_1, \ldots, f_m : T \to \mathbb{C}$ such that $\sum f_i v_i = w_c$. In particular, $\sum f_i(t) v_i(t) = w$. Let $\nu_i = f_i(t)$ and consider the element $\sum \nu_i v_i \in V$. At $t$, this evaluates to $\sum \nu_i v_i(t) = \sum f_i(t) v_i(t) = w$. Hence $e_t : V \to \mathbb{C}^n$ is surjective for $t \in T$.

Thus $C^\infty(T, \mathbb{C}) \otimes V \to C^\infty(T, \mathbb{C}^n)$ is surjective if and only if $e_t : V \to \mathbb{C}^n$ is surjective for all $t \in T$. Hence $V$ is almost generating if and only if there is a dense set $T \subseteq S^1$, which we may assume to be open, such that $e_t : V \to \mathbb{C}^n$ is surjective for all $t \in T$. $V$ is generating if and only if we are able to take $T = S^1$. □

**Proposition 3.10.** *Let $V \subseteq L\mathbb{C}^n$ be a finite dimensional subspace. Let $A_V$ be its stabiliser algebra. Then $A_V$ is finite dimensional if and only if $V$ is almost generating.*

*Proof.* Because $A_V$ is the stabiliser algebra of $V$, there is an algebra homomorphism $A_V \to \mathrm{End}(V)$. If this is injective, this demonstrates that $A_V$ is finite dimensional. If it is not injective, we shall show that its kernel contains a subalgebra isomorphic to $\Omega_\flat\mathbb{C}$, the algebra of base point preserving smooth functions on $S^1$ which are infinitely flat at 1.

Suppose that $V$ is almost generating and let $a \in A_V$ lie in the kernel of $A_V \to \mathrm{End}(V)$. Thus $av = 0$ for all $v \in V$. Let $T \subseteq S^1$ be as in the definition of almost generating for $V$. Let $t \in T$ and $w \in \mathbb{C}^n$. By proposition 3.9, there is some $v \in V$ such that $v(t) = w$. Hence $0 = a(t)v(t) = a(t)w$. As $w$ is arbitrary in $\mathbb{C}^n$, $a(t) = 0$. The continuity of $a$ combined with the density of $T$ in $S^1$ now show that $a = 0$. Hence $A_V \to \mathrm{End}(V)$ is injective.

Suppose that $V$ is not almost generating. Let $T \subseteq S^1$ be the open set on which $\dim e_t(V) = n$. As $V$ is not almost generating, $\overline{T} \neq S^1$. Let $l$ be the maximum value of $\dim e_t(V)$ on $S^1 \setminus \overline{T}$. Note that this is attained. From the proof of proposition 3.9, the set of $t \in S^1 \setminus \overline{T}$ where $\dim e_t(V) = l$ is open. Choose $t_0 \in S^1 \setminus \overline{T}$ such that $\dim e_{t_0}(v) = l$.

There are vectors $v_1, \ldots, v_l \in V$ such that $\{v_1(t_0), \ldots, v_l(t_0)\}$ is a linearly independent set at $t_0$. Extend this to a basis for $\mathbb{C}^n$ by adding $w_{n-l}, \ldots, w_n$. Then there is an open connected neighbourhood $U$ of $t_0$ for which the set:

$$\{v_1(t), \ldots, v_l(t), w_{n-l}, \ldots, w_n\}$$

is a basis for $\mathbb{C}^n$ for $t \in U$ with $e_t(V)$ spanned by $\{v_1(t), \ldots, v_l(t)\}$.

Let $\Omega_U\mathbb{C}$ denote the space of smooth $\mathbb{C}$ valued functions on $S^1$ with support in $U$. If $U = S^1$, let $\Omega_U\mathbb{C} = \Omega_\flat\mathbb{C}$.[1] If $U \neq S^1$, as $U$ is connected it is diffeomorphic to $(0, 1)$ and such a diffeomorphism defines in turn a diffeomorphism $\Omega_U\mathbb{C} \cong \Omega_\flat\mathbb{C}$.

For $t \in U$, let $P_t : \mathbb{C}^n \to \mathbb{C}^n$ be the projection onto $\langle w_n \rangle$ with kernel spanned by $\{v_1(t), \ldots, v_l(t), w_{n-l}, w_{n-1}\}$. Define an action of $\Omega_U\mathbb{C}$ on $L\mathbb{C}^n$ by $f \cdot w(t) = f(t)P_t w(t)$ for $t \in U$ and zero elsewhere. That $f$ is zero outside $U$ shows that this is well-defined.

By construction, for $v \in V$ and $t \in U$, $P_t v(t) = 0$ and hence $f \cdot v = 0$. Thus $f \in A_V$ and moreover, $f \in \ker A_V \to \mathrm{End}(V)$. Thus $\ker A_V \to \mathrm{End}(V)$ contains a subalgebra isomorphic to $\Omega_\flat\mathbb{C}$.

It is possible to extend this argument to determine $\ker A_V \to \mathrm{End}(V)$ precisely, but the above is sufficient to prove the proposition. □

## 3.3 Compact Lie Groups

In this section, we show that every compact Lie group acting on $L\mathbb{C}^n$ through a representation $\rho : G \to LU_n$ has an invariant generating subspace. From this,

[1] This is linguistic again and is purely to avoid having to treat the two cases separately; if $U$ were $S^1$, $L\mathbb{C}$ would do.

theorem 1.2 part 2 follows directly. Rather than considering the action of $G$ on $L\mathbb{C}^n$, we consider the induced action of $G$ on $LM_n$. We wish to find vectors in $L\mathbb{C}^n$ that generate a finite dimensional $G$-invariant generating subspace of $L\mathbb{C}^n$ and considering $LM_n$ allows us to find these vectors all in one go.

We shall use aspects of the Peter-Weyl theorem for representations of compact Lie groups. As usually stated, the Peter-Weyl theorem deals with unitary representations on Hilbert spaces. This is not readily applicable to our situation. However, the proof of the Peter-Weyl theorem actually applies in a stronger case which is sufficient for our purposes. We refer to [Kna98] for the necessary machinery. In [Kna98], the proof of the Peter-Weyl theorem is only given for unitary groups. Thus to derive the necessary extension for all compact Lie groups, we first use the Peter-Weyl theorem to deduce that all compact Lie groups are isomorphic to some subgroup of a unitary group and then use the proof of the Peter-Weyl theorem in [Kna98] to deduce the strengthening we require.

The following definitions are from [Kna98, ch III, §I]:

**Definition 3.11.** 1. A *matrix coefficient* of $G$ is a function $G \to \mathbb{C}$ defined by $g \to \langle \psi(g)u, v\rangle$, where $\psi : G \to U(V)$ is a unitary representation of $G$ on a finite dimensional vector space $V$ and $u, v \in V$ are fixed elements.

2. Let $\psi : G \to Gl(X)$ be a representation of $G$ on a Banach space $X$. For $f : G \to \mathbb{C}$ continuous, define $\psi(f) : X \to X$ by:

$$\psi(f)x = \int_G f(g)\psi(g)x d\mu$$

The map $C(G) \times X \to X$, $(f, x) \to \psi(f)x$, is linear in both variables and is continuous for the product topology. In fact, it is continuous as a map $L^1(G) \times X \to X$ as the following estimate shows:

$$\begin{aligned}
\|\psi(f)x\| &= \left\| \int_G f(g)\psi(g)x d\mu \right\| \\
&\leq \int_G |f(g)| \, \|\psi(g)\| \, \|x\| \, d\mu \\
&\leq \|x\| \int_G |f(g)| \, d\mu \sup_{g\in G} \|\psi(g)\| \\
&\leq \|x\| \, \|f\|_1 \, \|G\|
\end{aligned}$$

where $\|G\| := \sup_{g\in G} \|\psi(g)\|$. This exists by virtue of the facts that $G$ is compact and the map $G \to Gl(X) \to \mathbb{R}$, given by $g \to \|\psi(g)\|$, is continuous.

[Kna98, Theorem 1.14] states that every finite dimensional representation of a compact Lie group is smooth. As a corollary of this we obtain:

**Corollary 3.12.** *All matrix coefficients of $G$ are smooth functions on $G$.*

Using the technique illustrated in the proof of the Peter-Weyl theorem in [Kna98, theorem 3.7(a)], we find that:

**Proposition 3.13.** *The linear span of the matrix coefficients of $G$ is a uniformly dense subspace of $C(G)$, the space of continuous $\mathbb{C}$-valued functions on $G$.*

*Proof.* If $\phi : G \to H$ is a Lie group homomorphism, any matrix coefficient of $H$ defines one on $G$. Therefore, considering $G$ as a subgroup of some $U_m$, the linear span of the matrix coefficients of $U_m$ is contained within the linear span of the matrix coefficients of $G$. The restriction map $C(U_m) \to C(G)$ is a continuous, open map and therefore as the linear span of matrix coefficients of $U_m$ is uniformly dense in $C(U_m)$, its image is uniformly dense in $C(G)$. Hence the linear span of the matrix coefficients of $G$ is uniformly dense in $C(G)$. □

Let $\rho : G \to LGl_n$ be a Lie group homomorphism and consider the induced action of $G$ on $LM_n$. The inclusion $LGl_n \to L_{\text{cts}}Gl_n$ defines a representation of $G$ on $L_{\text{cts}}M_n$ and thus defines $C(G) \times L_{\text{cts}}M_n \to L_{\text{cts}}M_n$ as above. Moreover, the restriction to the subspaces of smooth maps on the left defines a continuous, bilinear map $C^\infty(G) \times LM_n \to LM_n$. In other words, if it so happens that $f : G \to \mathbb{C}$ and $\alpha : S^1 \to M_n$ are smooth then:

$$t \to \int_G f(g)\rho(g)\alpha(t)d\mu$$

is a smooth map, $S^1 \to M_n$.

The proof of [Kna98, theorem 3.7(d)] together with the above yields a proof of the following:

**Proposition 3.14.** *For any $\alpha \in LM_n$ and $\epsilon > 0$ there is some $\beta \in LM_n$ for which $\|\alpha - \beta\|_\infty < \epsilon$ and $\beta$ lies in a $G$-invariant, finite dimensional subspace of $LM_n$.*

We are regarding $M_n$ as the (finite dimensional) Banach algebra of operators on $\mathbb{C}^n$ and equip it with the operator norm, corresponding to the standard norm on $\mathbb{C}^n$. This in turn defines the norm $\|\cdot\|_\infty$ on $L_{\text{cts}}M_n$.

*Proof.* The method of the proof is to express $\beta$ as $\rho(h)\alpha$ for a suitable choice of $h$. We start in the continuous case and then use the remark above to deduce that if $\alpha$ and $h$ were originally smooth, $\beta$ is smooth. Let $\|G\|_\infty = \sup_{g\in G}\{\|\rho(g)\|_\infty\}$.

As in the proof of [Kna98, theorem 3.7(d)], we note that if $h \in C(G)$ is a (finite) linear combination of matrix coefficients, $\rho(h)\alpha$ lies in a $G$-invariant, finite dimensional subspace of $LM_n$.

For an open neighbourhood $U$ of $1 \in G$, let $f_U : G \to \mathbb{C}$ be a positive, continuous function with support in $U$, $\|f_U\|_1 = 1$. Then:

$$\begin{aligned}\rho(f_U)\alpha - \alpha &= \int_G f_U(g)\rho(g)\alpha d\mu - \|f_U\|_1 \alpha \\ &= \int_G f_U(g)\rho(g)\alpha d\mu - \int_G |f_U(g)|\, d\mu\alpha \\ &= \int_G f_U(g)\rho(g)\alpha d\mu - \int_G f_U(g)\alpha d\mu\end{aligned}$$

$$
\begin{aligned}
&= \int_G f_U(g)\,(\rho(g)-1)\,\alpha d\mu \\
\text{so } \|\rho(f_U)\alpha-\alpha\|_\infty &\le \int_G |f_U(g)|\,\|\rho(g)-1\|_\infty\,\|\alpha\|_\infty\,d\mu \\
&= \int_U |f_U(g)|\,\|\rho(g)-1\|_\infty\,\|\alpha\|_\infty\,d\mu \\
&\le \|\alpha\|_\infty \sup_{g\in U}\|\rho(g)-1\|_\infty
\end{aligned}
$$

Thus we can find $U_0$ sufficiently small for which $f_0 := f_{U_0}$ satisfies the inequality $\|\rho(f_0)\alpha-\alpha\|_\infty < \epsilon/2$. As $f_0 \in C(G)$, there is some linear combination of matrix coefficients, $h$, such that $\|f_0-h\|_\infty < \epsilon/(2\,\|G\|_\infty\,\|\alpha\|_\infty)$. Thus:

$$
\begin{aligned}
\|\rho(h)\alpha-\alpha\|_\infty &= \|\rho(h)\alpha-\rho(f_0)\alpha+\rho(f_0)\alpha-\alpha\|_\infty \\
&\le \|\rho(h)\alpha-\rho(f_0)\alpha\|_\infty + \|\rho(f_0)\alpha-\alpha\|_\infty \\
&< \|h-f_0\|_\infty\,\|\alpha\|_\infty\,\|G\|_\infty + \epsilon/2 \\
&< \epsilon
\end{aligned}
$$

Set $\beta = \rho(h)\alpha$, then $\beta$ satisfies the conditions of the proposition for the continuous case. Moreover, as $h$ is a linear combination of matrix coefficients it is smooth. Hence if $\alpha$ is smooth, $\beta$ is also smooth. □

**Theorem 3.15.** *Let $\rho : G \to LGl_n$ be a Lie group homomorphism with $G$ compact. Then there exists a $G$-invariant, generating subspace of $L\mathbb{C}^n$.*

*Proof.* From proposition 3.14 we know that there exists $\beta \in LM_n$ lying in a $G$-invariant, finite dimensional subspace of $LM_n$ such that $\|1-\beta\|_\infty < 1$. The condition $\|1-\beta\|_\infty < 1$ implies that $\beta(t)$ is invertible for all $t$ and hence the columns of $\beta(t)$ form a basis for $\mathbb{C}^n$ for each $t$.

Because $\beta$ lies in a $G$-invariant, finite dimensional subspace of $LM_n$, the loops $t \to \beta(t)x_i$ each lie in a $G$-invariant, finite dimensional subspace of $L\mathbb{C}^n$, where $\{x_1,\dots,x_n\}$ is the standard basis for $\mathbb{C}^n$. Thus there is a $G$-invariant, finite dimensional subspace $V$ of $L\mathbb{C}^n$ which contains the loops defined by the columns of $\beta$. As evaluation at $t$ maps these vectors to a basis for $\mathbb{C}^n$, the evaluation map $e_t : V \to \mathbb{C}^n$ is surjective for all $t$. Hence $V$ is generating. □

**Corollary 3.16.** *Let $\rho : G \to LU_n$ be a homomorphism of Lie groups with $G$ compact. Then there is a homomorphism $\tilde{\rho} : G \to U_n$ and an element $\gamma \in LU_n$ such that $\rho = c_\gamma\tilde{\rho}$.*

This, together with corollary 3.3, yields a proof of theorem 1.2, part 2. It also proves theorem 3.1 as follows:

*Proof of theorem 3.1.* If $G$ is a compact subgroup of either $LU_n$ or $\Omega U_n$, there is a homomorphism of Lie groups $\rho : G \to LU_n$. Thus there is a conjugation in $LU_n$ which takes $G$ to a subgroup of $U_n$. If $G$ were originally in $\Omega U_n$, conjugation would not change that, so it is conjugate to a subgroup of $\Omega U_n \cap U_n = \{1\}$. Hence $\Omega U_n$ has no compact subgroups and those of $LU_n$ are conjugate to subgroups of $U_n$. □

***Acknowledgements:*** This work began as an elaboration of the work in [CS02]. The author would like to thank Ralph Cohen for helpful conversations. The author would also like to thank Persi Diaconis and Dan Bump for their help at stages of the work.

# References

[Bro78] Robert F. Brown. A homological characterization of inner automorphisms of compact Lie groups. In Kenneth C. Millett, editor, *Proceedings of a Symposium held at the University of California, Santa Barbara, Calif., in honor of Raymond L. Wilder, July 25–29, 1977*, pages 38–42. Springer-Verlag, Berlin, 1978. Lecture Notes in Mathematics, 664.

[CS02] R. L. Cohen and A. Stacey. Fourier decompositions of loop bundles, math.AT/02010351, 2002.

[Dyn52] E. B. Dynkin. A connection between homologies of a compact Lie group and its subgroups. *Doklady Akad. Nauk SSSR (N.S.)*, 87:333–336, 1952.

[EE70] J. Eells and K. D. Elworthy. On Fredholm manifolds. *Actes du Congrés International des Mathématiciens*, 2:215–230, 1970.

[Elw68] K. Elworthy. Fredholm maps and $gl_c(E)$-structures, (Oxford thesis). *Bull. Amer. Math. Soc.*, 74:582–586, 1968.

[Gel91] G. Gelbrich. On pointwise conjugate homomorphisms of compact Lie groups. *Seminar Sophus Lie*, 1:217–223, 1991.

[Kna98] A. Knapp. *Lie Groups, Lie Algebras, and Cohomology.* Number 34 in Mathematical Notes. Princeton University Press, Princeton, New Jersey, first edition, 1998.

[Kui65] N. H. Kuiper. The homotopy type of the unitary group of Hilbert space. *Topology*, 3:19–30, 1965.

[Mor01] J. Morava. The tangent bundle of an almost-complex free loopspace, math.DG/0108085, 2001.

[Wit88] E. Witten. *The Index of the Dirac Operator in Loop Space*, volume 1326 of *Lecture Notes in Mathematics*, pages 161–181. Springer-Verlag, 1988.