Constructing Smooth Loop Spaces

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Abstract

We consider the general problem of constructing the structure of a smooth manifold on a given space of loops in a smooth finite dimensional manifold. By generalising the standard construction for smooth loops, we derive a list of conditions for the model space which, if satisfied, mean that a smooth structure exists.

We also show how various desired properties can be derived from the model space; for example, topological properties such as paracompactness. We pay particular attention to the fact that the loop spaces that can be defined in this way are all homotopy equivalent; and also to the action of the circle by rigid rotations.

1 Introduction

It is often convenient to regard a space of certain loops in a smooth manifold as a smooth manifold itself with the aim of doing differential topology thereon. Depending on the application this approach can vary from the conceptual to the rigorous. The two most popular types of loop are continuous and smooth, for both of which there is a rigorous theory of infinite dimensional manifolds making these into smooth manifolds: [Kli95], [Lan85], [Mil84], [Omo97]. Other types of loop have also been considered: it is often convenient to use a manifold modelled on a Hilbert space whence one usually uses the space of loops with square-integrable first derivative.

There is a standard method of constructing the smooth structure which is used in each case mentioned above. Our main purpose in this paper is to extend this construction to a reasonably arbitrary class of loops. In so doing, we obtain a list of conditions on the model space such that if they hold then this method applies. This enables us to reduce the general problem of whether or not a particular type of loop forms a smooth manifold to a check-list for the model space and means that we can avoid writing out the full construction in each and every case.

Before giving the list of conditions, we feel it relevant to comment on calculus. The examples of spaces given in the first paragraph were all "nice" as regards calculus. Two were Banach spaces (one a Hilbert space, no less) and the other is one of the nicest Fréchet spaces that one could hope to meet. In all of these cases, calculus is well-understood and well-defined. However, once one leaves the realm of Fréchet spaces and departs for more general locally convex topological spaces then the notion of what is "smooth" becomes increasingly

hard to pin down. The usual idea of taking "smooth" to mean "infinitely differentiable" leads to many complications, not least that this is not uniquely defineable. Fortunately, an alternative approach has been developed in which the notion of "smooth" is based on something other than differentiability. This calculus is laid out in the weighty tome [KM97]. The introduction and the historical remarks at the end of chapter 1 of [KM97] make for an interesting read on this subject.

The impact that this has on our work is more subtle than might be expected. The place where one would expect this issue to arise is in showing that the transition functions are smooth. However, this depends on certain functions on the model space being smooth and so we build this into one of our conditions. We can therefore phrase the corresponding condition in such terms that it could apply whatever type of calculus we were using.

There are other places, however, where the calculus used makes an appearance. The most important being the definition of an infinite dimensional manifold. One of the issues in infinite dimensional calculus is that maps can be smooth without being continuous and this leads to a certain *laissez faire* attitude to topology. The traditional definition of a manifold is of a topological space with a smooth atlas. The definition in [KM97] is of a *set* with a smooth atlas which is then topologised using said atlas. Thus if one wishes to apply the results of this paper using a calculus other than that of [KM97], this issue might be important. Certainly, the traditional approach to building infinite dimensional manifolds modelled on Banach spaces has been to mirror the standard approach of topology first and smooth structure second. Therefore, if taking a different calculus, it might be necessary to add the additional step of checking that the original topology and the new topology were one and the same. We shall not bother with this issue explicitly, but shall provide some tools which will help if it is considered important by others.

Having made that point, we turn to our list of conditions. We start with a class of maps $S^1 \to \mathbb{R}$ which we write as $L^x\mathbb{R}$, or as $C^x(S^1, \mathbb{R})$ when we want to emphasise the domain, and refer to as C^x–loops. We want to consider actual maps, and not equivalence classes of maps, because we want to be able to "locate" our maps on a manifold in order to be able to examine them in charts. Thus we regard $L^x\mathbb{R}$ as a subset of $\mathrm{Map}(S^1, \mathbb{R})$. Using the natural identification of $\mathrm{Map}(S^1, \mathbb{R}^n)$ with $\mathrm{Map}(S^1, \mathbb{R})^n$ we define $L^x\mathbb{R}^n$ as $(L^x\mathbb{R})^n$. For a subset $A \subseteq \mathbb{R}^n$, we define L^xA as the subset of $L^x\mathbb{R}^n$ consisting of maps which take values in A.

Our conditions are:

1. Being in $L^x\mathbb{R}$ is a *local* property.

   That is, for a loop $\gamma\colon S^1 \to \mathbb{R}$, then $\gamma \in L^x\mathbb{R}$ if there exists an open cover $\mathcal{U}$ of S^1 and loops $\gamma_U \in L^x\mathbb{R}$ for $U \in \mathcal{U}$ such that γ agrees with γ_U on U.

2. The set $L^x\mathbb{R}$ is a subspace of $\mathrm{Map}(S^1, \mathbb{R})$.

3. The vector space $L^x\mathbb{R}$ can be given a topology with respect to which it is a locally convex topological vector space.

4. The locally convex topological vector space $L^x\mathbb{R}$ is *convenient*.

   This is a completeness condition. We have phrased it in the language of [KM97] but it is the same as a concept known as *locally complete* which is

from ordinary functional analysis. Local completeness is weaker than sequential completeness, though it coincides with completeness for metrisable spaces. This completeness condition is to ensure that derivatives that ought to exist actually do.

5. As subspaces of $\mathrm{Map}(S^1, \mathbb{R})$ we have inclusions:

$$L\mathbb{R} \subseteq L^x\mathbb{R} \subseteq L^0\mathbb{R}$$

where $L\mathbb{R} = C^\infty(S^1, \mathbb{R})$ and $L^0\mathbb{R} = C^0(S^1, \mathbb{R})$. The inclusion maps are all continuous when each is given its standard topology.

We will be considering loops in a smooth manifold and therefore will need to know that the condition of being a C^x–loop is invariant under post-composition by diffeomorphisms. This essentially forces smooth loops to be C^x–loops. For the other inclusion, as we remarked above we want to be able to "locate" a loop on a manifold so that we can consider it in charts. The simplest way to do this is to ensure that a C^x–loop is continuous.

6. The action of post-composition of C^x–loops by smooth maps is well-defined and is smooth. That is, let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets; let $\phi\colon U \to V$ be a smooth map. The induced map $\phi_*\colon L^xU \to L^xV, \gamma \mapsto \phi \circ \gamma$, is well-defined and is smooth.

This is the crucial condition that will ensure that the transition functions are defined and are diffeomorphisms.

Having stated our conditions, we can now state our first theorem. We make two assumptions on our target manifold: that it be orientable and that it have no boundary. The first is really a convenience to allow us not to have to discuss twisted model spaces; the second is necessary as the loop space of a manifold with boundary is a complicated object indeed.

Theorem A *Let $L^x\mathbb{R}$ be a class of maps satisfying the above conditions. Let M be a smooth, orientable finite dimensional manifold without boundary. Then L^xM can be defined and is a smooth manifold in the sense of [KM97].*

We emphasise that the phrase "in the sense of [KM97]" does not refer to the calculus but to the definition of a smooth manifold once one has decided on a calculus.

Having defined the smooth structure, the next question is to examine the general properties of the manifold. In most cases, these descend from the model space. The result which allows us to devolve these properties is the following theorem on submanifolds.

Theorem B *Let $L^x\mathbb{R}$ be a class of maps satisfiying the above conditions. Let M, N be smooth, orientable finite dimensional manifolds without boundary and suppose that there is an embedding of M as a submanifold of N. Then L^xM is an embedded submanifold of L^xN. Moreover, if M is closed, resp. open, in N then L^xM is closed, resp. open, in L^xN.*

Corollary C *In the statement of theorem B suppose that we can take $N = \mathbb{R}^n$ with M closed in N. Then the following properties are inherited by L^xM from $L^x\mathbb{R}$: separable, metrisable, Lindelöf, paracompact, normal, smoothly regular, smoothly paracompact, and smoothly normal.*

That is, those properties which hold for $L^x\mathbb{R}$ also hold for L^xM.

The last property that we wish to examine is the natural circle action, and more generally the natural action of the diffeomorphisms of the circle.

Theorem D *Under the conditions of corollary C, the actions of the circle and of the diffeomorphisms of the circle are also inherited by L^xM from $L^x\mathbb{R}$.*

In light of this, we conclude this paper with a discussion as to the various possible levels of continuity and smoothness of the circle acting on an infinite dimensional locally convex topological vector space (lctvs).

This paper is organised as follows. In section 2 we prove some preliminary results, including the definition of L^xM. Section 3 is concerned with the construction of the charts and showing that the transition maps are diffeomorphisms; in particular we prove theorem A. In section 4 we transfer our attention to the topology of the manifold and prove theorem B and its corollaries. Finally, in section 5 we look at the action of the circle and its diffeomorphisms.

The standard construction of the smooth structure on the space of smooth loops can be found in many places, for example in [Mic80] and in [KM97]. Some other articles and books which deal with the infinite dimensional manifolds in varying levels of generality are: [Mil84], [Omo97], [Eel66], [EE71], [Lan85], [Kli95]. Most of the work in this paper is firmly in the realms of differential topology and should be comprehensible to anyone with a firm grasp of the basics of the theory in finite dimensions. The exception to this is the discussion of actions of the circle and diffeomorphism group which uses some standard functional analysis. This may be unfamiliar to differential topologists, at whom this article is aimed, in which case we recommend [Sch71] and [Jar81] for the necessary background.

We regard the circle as the quotient $\mathbb{R}/\mathbb{Z}$ and shall write it additively. We shall often write a small neighbourhood of a point as $(t-\epsilon, t+\epsilon)$ without worrying about "wrap-around"; either the "wrap-around" will have no effect on the subsequent discussion or we will be allowed to take ϵ small enough that there is no "wrap-around". We shall also employ the language of *intervals* for connected subsets of S^1 – including S^1 itself.

2 Preliminaries

In this section we shall set up the basic machinery that we need to define and construct the smooth manifold of loops. From hereon, let us assume that we have a class of loops, $L^x\mathbb{R}$, satisfying the conditions stated in the introduction. Let M be a smooth, orientable, finite dimensional manifold without boundary. Our first task is to define L^xM. Our second is to define and examine the space of C^x–sections in a smooth vector bundle over S^1; these will prove crucial in the atlas for L^xM.

2.1 Loops in a Manifold

To define L^xM we need to show that we can examine a loop locally to see whether or not it is in L^xM. Condition 1 is almost what we need but isn't quite local enough.

Definition 2.1 *Let $I \subseteq S^1$ be an open interval. Define $C^x(I, \mathbb{R})$ to be the space of maps $\gamma\colon I \to \mathbb{R}$ which are locally of type C^x. That is, there is an open cover $\mathcal{U}$ of I and maps $\gamma_U \in L^x\mathbb{R}$ for $U \in \mathcal{U}$ such that γ agrees with γ_U on U.*

Note that we don't assume that the whole function extends, merely that it locally extends. It follows from the definition that the restriction map $C^x(I, \mathbb{R}) \to C^x(J, \mathbb{R})$ is defined for $J \subseteq I$.

Lemma 2.2 *In the locality condition, it is enough to assume that the local functions are only defined locally.*

That is, a map $\gamma\colon S^1 \to \mathbb{R}$ is a C^x–map if there is a cover $\mathcal{U}$ of S^1 by open intervals and functions $\gamma_U \in C^x(U, \mathbb{R})$ such that γ agrees with γ_U on U.

Proof. Let $t \in S^1$. Then there is some $U \in \mathcal{U}$ with $t \in U$. As $\gamma_U \in C^x(U, \mathbb{R})$ there is an open cover $\mathcal{V}$ of U and loops $\beta_V \in L^x\mathbb{R}$ such that γ_U agrees with β_V on V. There is some $V \in \mathcal{V}$ such that $t \in V$. Then on V, β_V agrees with γ_U which agrees with γ. Repeating this for all $t \in S^1$ gives the family required to apply condition 1. □

Another piece of preliminary work that we need to do, or rather just to note as it is trivial, is to show that all of our conditions and results are equally valid for $\mathbb{R}^n$ as for $\mathbb{R}$. Condition 6 is already in full generality.

Lemma 2.3 *Let $L^x\mathbb{R}$ be a class of maps satisfying the conditions of section 1. Then $L^x\mathbb{R}^n$ satisfies analogous conditions and the corresponding version of lemma 2.2.*

Proof. This is trivial and follows from the fact that $L^x\mathbb{R}^n$ is canonically identified with $(L^x\mathbb{R})^n$. □

One other result that we need, which is equally trivial, is the following statement about open sets.

Lemma 2.4 *Let $U \subseteq \mathbb{R}^n$ be open. Then L^xU is open in $L^x\mathbb{R}^n$.*

Proof. This holds for $L^0\mathbb{R}^n$ and so holds because $L^xU = L^0U \cap L^x\mathbb{R}^n$. □

With this in place we can define our space of interest.

Definition 2.5 *Let $L^x\mathbb{R}$ be a class of maps satisfying the conditions of section 1. Let M be a smooth finite dimensional manifold. Define L^xM to be the subset of* $\mathrm{Map}(S^1, M)$ *consisting of those loops $\gamma\colon S^1 \to M$ for which there exists a covering $\{I_\alpha\colon \alpha \in A\}$ of S^1 by open intervals and charts $\{(\iota_\alpha, U_\alpha, V_\alpha) : \alpha \in A\}$ for M (not necessarily making a full atlas) such that for each $\alpha \in A$, $\gamma(I_\alpha) \subseteq V_\alpha$ and the map:*

$$\gamma_\alpha\colon I_\alpha \to S^1 \xrightarrow{\gamma} V_\alpha \xrightarrow{\iota_\alpha^{-1}} U_\alpha$$

lies in $C^x(I_\alpha, U_\alpha)$.

Thus we have defined C^x–loops in M to be those that look like C^x–loops whenever we look locally. Lemma 2.2 and condition 6 easily combine to show that this definition does not depend on any of the choices made.

There is another way to make this definition; if M were a submanifold of, say, $\mathbb{R}^n$ then we already have a definition of L^xM: namely that subset of $L^x\mathbb{R}^n$ of loops which take values in M. The next result shows that these two definitions coincide. We prefer the above as the actual definition as it treats the manifold without reference to any surrounding space.

Proposition 2.6 *Let $L^x\mathbb{R}$ be a class of maps satisfying the conditions in section 1. Let M, N be smooth, finite dimensional manifolds with M an embedded submanifold of N. Then*

$$L^xM = \{\gamma \in L^xN : \gamma(S^1) \subseteq M\}.$$

Proof. Since this is true for arbitrary maps, what we need to show is that a loop in M is a C^x–loop when viewed in M if and only if it is a C^x–loop when viewed in N. To do this, we ensure that the charts in M in which we are looking are all submanifold charts; that is, restrictions of charts on N which take M to $\mathbb{R}^k$ inside $\mathbb{R}^n$. The desired result then follows from the fact that a loop in $\mathbb{R}^n$ is a C^x–loop if and only if its co-ordinates are C^x–loops; whereupon the C^x–loops in $\mathbb{R}^k$ are precisely the C^x–loops in $\mathbb{R}^n$ which happen to lie in $\mathbb{R}^k$. □

2.2 Sections and Submanifolds

In this section we define and examine the space of C^x–sections of a vector bundle over S^1.

Definition 2.7 *Let $E \to S^1$ be a smooth fibre bundle. Define $\Gamma^x_{S^1}(E)$ as the space of sections of E which are C^x–loops when viewed as maps into the total space of E.*

In the particular case that E is an orientable vector bundle, we can identify this space of sections with $L^x\mathbb{R}^n$.

Lemma 2.8 *Let $E \to S^1$ be a smooth orientable vector bundle of fibre dimension n. A smooth trivialisation of E defines a bijection $\Gamma^x_{S^1}(E) \to L^x\mathbb{R}^n$. The map $L^x\mathbb{R}^n \to L^x\mathbb{R}^n$ induced by two trivialisations of E is a linear diffeomorphism.*

Proof. As it is obvious that a smooth trivialisation of E defines a bijection from the space of all sections of E to $\mathrm{Map}(S^1, \mathbb{R}^n)$ all we need to check to show the first part is that a section of E is a C^x–section if and only if this bijection takes it to a C^x–loop in $\mathbb{R}^n$.

Condition 6 assures us that a diffeomorphism on the target space induces a bijection on the spaces of C^x–loops. Therefore we have a bijection from L^xE to $L^x(S^1 \times \mathbb{R}^n)$. As the trivialisation of E intertwines the projection maps, this bijection takes sections to sections and so induces a bijection $\Gamma^x_{S^1}(E) \to \Gamma^x_{S^1}(S^1 \times \mathbb{R}^n)$. Thus the problem is reduced to the case of a trivial vector bundle.

Now it is clear from the definition of C^x–loops in a manifold that a loop in a (finite) product is a C^x–loop if and only if each of the factors is a C^x–loop. Therefore a loop in $S^1 \times \mathbb{R}^n$ is a C^x–loop if and only if the projections to S^1 and to $\mathbb{R}^n$ are C^x–loops. Now a loop is a section if and only if it projects to the identity on S^1 and the identity is smooth, whence a C^x–loop. Therefore a section of $S^1 \times \mathbb{R}^n$ is a C^x–section if and only if the projection to $\mathbb{R}^n$ produces a C^x–loop.

Tracing this through shows that the trivialisation does induce a bijection $\Gamma^x_{S^1}(E) \to L^x\mathbb{R}^n$.

Two such trivialisations of E define a diffeomorphism $\phi\colon S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n$ covering the identity on S^1. We extend this to a smooth map $\mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}^2 \times \mathbb{R}^n$, viewing S^1 as a submanifold of $\mathbb{R}^2$. Note that we do not assume that this extension is a diffeomorphism (an extension to a diffeomorphism may not

exist). The induced map $L^x\mathbb{R}^n \to L^x\mathbb{R}^n$ factors as:

$$\begin{aligned}
L^x\mathbb{R}^n &\to L^x(\mathbb{R}^2 \times \mathbb{R}^n) & \gamma &\mapsto (0, \gamma) \\
L^x(\mathbb{R}^2 \times \mathbb{R}^n) &\to L^x(\mathbb{R}^2 \times \mathbb{R}^n) & (\alpha, \gamma) &\mapsto (\alpha + \iota, \gamma) \\
L^x(\mathbb{R}^2 \times \mathbb{R}^n) &\xrightarrow{\phi_*} L^x(\mathbb{R}^2 \times \mathbb{R}^n) & (\alpha, \gamma) &\mapsto \phi \circ (\alpha, \gamma) \\
L^x(\mathbb{R}^2 \times \mathbb{R}^n) &\to L^x\mathbb{R}^n & (\alpha, \gamma) &\mapsto \gamma,
\end{aligned}$$

where $\iota\colon S^1 \to \mathbb{R}^2$ is the inclusion. The first map is continuous and linear, hence smooth. The second map is a translation, hence smooth. The third map is smooth by condition 6. The final map is continuous and linear, hence smooth. Thus $\phi_*\colon L^x\mathbb{R}^n \to L^x\mathbb{R}^n$ is smooth. Applying the same to ϕ^{-1} shows that ϕ_* is a diffeomorphism, as required. □

Using this we transfer the smooth structure of $L^x\mathbb{R}^n$ to $\Gamma^x_{S^1}(E)$. If we are using the calculus of [KM97], it is a curious fact that although ϕ_* is a linear diffeomorphism, it need not be a homeomorphism as it, or its inverse, need not be continuous. They will, however, be *bounded* maps. If we assume that the topology on $L^x\mathbb{R}^n$ is *bornological* – a condition that we can readily impose by a standard alteration of the topology – then bounded maps are continuous and so we do have a homeomorphism.

Corollary 2.9 *Let $E \to S^1$ be a finite dimensional orientable smooth vector bundle. Then $\Gamma^x_{S^1}(E)$ naturally has the structure of a convenient vector space and is diffeomorphic to $L^x\mathbb{R}^n$, where $n = \dim E$.* □

We can adapt the proof of lemma 2.8 to demonstrate the following properties of $\Gamma^x_{S^1}(E)$.

Lemma 2.10 *Let $E, F \to S^1$ be finite dimensional orientable smooth vector bundles. Let $U \subseteq E$ and $V \subseteq F$ be open subsets of the total space and $\phi\colon U \to V$ a smooth map covering the identity on S^1. Let $\Gamma^x_{S^1}(U) := \{\gamma \in \Gamma^x_{S^1}(E) : \gamma(S^1) \subseteq U\}$, and similarly for V. Then $\Gamma^x_{S^1}(U)$ is open in $\Gamma^x_{S^1}(E)$ and the induced map $\gamma \mapsto \phi \circ \gamma$ is a smooth map $\Gamma^x_{S^1}(U) \to \Gamma^x_{S^1}(V)$.*

Proof. It is sufficient to examine the case where E and F are trivial; say, $E = S^1 \times \mathbb{R}^m$ and $F = S^1 \times \mathbb{R}^n$. In this case we have a topological embedding of $L^x\mathbb{R}^m$ as an affine subspace of $L^x(\mathbb{R}^2 \times \mathbb{R}^m)$ via $\gamma \mapsto (\iota, \gamma)$. There is a set $W \subseteq \mathbb{R}^2 \times \mathbb{R}^m$ which restricts to U on $S^1 \times \mathbb{R}^m$ and, under the above embedding, $\Gamma^x_{S^1}(U)$ is the intersection of $L^x\mathbb{R}^m$ with L^xW, hence is open in $\Gamma^x_{S^1}(E)$.

Now smoothness is a local property so we may assume that $\phi\colon U \to V$ extends to a smooth map $\mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}^2 \times \mathbb{R}^n$. To deduce the general case from this we choose a sequence of open sets U_n such that $U = \bigcup U_n$ and $\overline{U}_n \subseteq U_{n+1}$. Using bump functions we can define maps $\phi_n\colon \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}^2 \times \mathbb{R}^n$ such that $\phi_n = \phi$ on U_n. Thereupon if we can show that each ϕ_n is smooth then we can deduce that ϕ is locally smooth and hence smooth.

We now use the same method as in the proof of lemma 2.8 to deduce that $\phi_*\colon \Gamma^x_{S^1}(U) \to \Gamma^x_{S^1}(V)$ is smooth. □

3 Building a Smooth Manifold

In this section we construct the charts for L^xM and show that the transition maps are smooth.

3.1 Charts

The key tool for defining the charts for the loop space is the notion of a *local addition* on M, cf [KM97, §42.4]:

Definition 3.1 *Let $U \subseteq M$ be an open subset of M. A* local addition over U *for M consists of an open subset $U \subseteq M$ and smooth map $\eta\colon TU \to U$ such that*

1. *the composition of η with the zero section is the identity on U, and*
2. *there exists an open neighbourhood V of the diagonal in U such that the map $\pi \times \eta\colon TU \to U \times U$ is a diffeomorphism onto V.*

For a subset $A \subseteq M$, a local addition for A *is a local addition defined over a neighbourhood of A. If $f\colon X \to M$ is a map, a* local addition for f *is a local addition defined over a neighbourhood of the image of f.*

This is closely related to what is called a local addition in [KM97, §42.4] but is not quite the same. One difference, that we use the whole of the fibres, is for simplicity whilst the other difference, that we do not assume it to be defined on the whole of M, is to make later analysis easier. The following result is contained in the discussion following the definition of a local addition in [KM97, §42.4]:

Proposition 3.2 *Any finite dimensional manifold without boundary admits a local addition defined over the whole of the manifold.* □

We start by constructing our chart maps for L^xM. They will be anchored at *smooth* loops rather than arbitrary elements of L^xM. This is to ensure that all the maps between finite dimensional objects are smooth so we don't need to consider C^x–maps with arbitrary domains.

Lemma 3.3 *Let $\alpha\colon S^1 \to M$ be a* smooth *loop. Let $\eta\colon TU \to U$ be a local addition for α with neighbourhood V of the diagonal. Define the set $U_\alpha \subseteq L^xM$ by:*

$$U_\alpha := \{\beta \in L^xM\colon (\alpha, \beta) \in L^xV\}.$$

Then $\pi \times \eta\colon TU \to V$ induces a bijection from $\Gamma^x_{S^1}(\alpha^*TM)$ to U_α. Under this bijection, the zero section of α^*TM maps to α.

Proof. By definition, the image of α lies in U. As U is an open subset of M, the bundles α^*TM and α^*TU are naturally identified. We claim that there is a diagram:

$$\begin{array}{ccc} L^xTU & \xrightarrow{(\pi\times\eta)_*} & L^xV \\ \uparrow & & \uparrow{\scriptstyle \beta\mapsto(\alpha,\beta)} \\ \Gamma^x_{S^1}(\alpha^*TM) & & U_\alpha, \end{array}$$

such that the map at the top is a bijection and takes the image of the left-hand vertical map to the image of the right-hand one. Both of the vertical maps are

injective – the right-hand one obviously so, we shall investigate the left-hand one in a moment – and thus the bijection $(\pi \times \eta)_*$ induces a bijection from the lower left to the lower right.

As TU is an open subset of TM and V of $M \times M$, both are smooth manifolds. The map $\pi \times \eta \colon TU \to V$ is a diffeomorphism and hence induces a bijection on the sets of C^x–maps into each. This is the map we have labelled $(\pi \times \eta)_*$.

The left-hand vertical map, $\Gamma^x_{S^1}(\alpha^*TM) \to L^xTU$, is defined as follows: the total space $\alpha^*TM = \alpha^*TU$ is:

$$\{(t, v) \in S^1 \times TU \colon \alpha(t) = \pi(v)\}.$$

It is an embedded submanifold of $S^1 \times TU$. Therefore by proposition 2.6, a map into α^*TM is a C^x–map if and only if the compositions with the projections to S^1 and to TU are both C^x–maps. Now a map $S^1 \to \alpha^*TU$ is a section if and only if it projects to the identity on S^1. Therefore there is a bijection (of sets):

$$\begin{aligned}\Gamma^x_{S^1}(\alpha^*TM) &\cong \{\beta \in L^xTU \colon (t, \beta(t)) \in \alpha^*TM \text{ for all } t \in S^1\} \\ &= \{\beta \in L^xTU \colon \alpha(t) = \pi\beta(t) \text{ for all } t \in S^1\} \\ &= \{\beta \in L^xTU \colon \pi_*\beta = \alpha\}.\end{aligned}$$

In particular, the map $\Gamma^x_{S^1}(\alpha^*TM) \to L^xTU$ is injective.

We apply $(\pi \times \eta)_*$ to the image of $\Gamma^x_{S^1}(\alpha^*TM)$ and see that it is the preimage under this map of everything of the form (α, γ) in L^xV. By construction, $\gamma \in L^xM$ is such that $(\alpha, \gamma) \in L^xV$ if and only if $\gamma \in U_\alpha$. Hence $(\pi \times \eta)_*$ identifies the image of $\Gamma^x_{S^1}(\alpha^*TM)$ with $\{\alpha\} \times U_\alpha$.

Finally, note that the zero section of α^*TM maps to the image of α under the zero section of TU. Since η composed with the zero section of TU is the identity on U, the image of the zero section of α^*TM in V is (α, α) as required which projects to α in U_α. □

Definition 3.4 *Let $\Psi_\alpha \colon \Gamma^x_{S^1}(\alpha^*TM) \to U_\alpha$ be the resulting bijection.*

In detail, this map is as follows: let $\beta \in \Gamma^x_{S^1}(\alpha^*TM)$ and let $\tilde{\beta}$ be the corresponding loop in TU, so $\beta(t) = (t, \tilde{\beta}(t))$. Then $(\pi \times \eta)_*(\tilde{\beta}) = (\alpha, \eta_*(\tilde{\beta}))$ so $\Psi_\alpha(\beta) = \eta_*(\tilde{\beta})$.

The domains of these charts are naturally convenient vector spaces. On the other end, we need to show that the codomains cover L^xM.

Lemma 3.5 *The codomains of the charts cover L^xM.*

Proof. This follows from the density of LM in L^0M. We choose a local addition defined over the whole of M, $\eta \colon TM \to M$, with corresponding neighbourhood $V \subseteq M \times M$ of the diagonal. As such, for any $\beta \in L^0M$ there is some $\alpha \in LM$ such that $(\alpha, \beta) \in L^0V$. Whereupon, if $\beta \in L^xM$ then $\beta \in U_\alpha$. Hence the sets U_α cover L^xM. □

3.2 Transitions

Having defined the charts, we turn to the transition functions. Let α_1, α_2 be smooth loops in M. Let $\eta_1 \colon TU_1 \to U_1$ and $\eta_2 \colon TU_2 \to U_2$ be local additions for α_1 and α_2 respectively, with corresponding open sets $V_1 \subseteq U_1 \times U_1$ and $V_2 \subseteq U_2 \times U_2$. Let $\Psi_1 \colon \Gamma^x_{S^1}({\alpha_1}^*TM) \to U_{\alpha_1}$, $\Psi_2 \colon \Gamma^x_{S^1}({\alpha_2}^*TM) \to U_{\alpha_2}$ be the corresponding charts. Let $U_{12} \coloneqq U_{\alpha_1} \cap U_{\alpha_2}$.

Lemma 3.6 *Let $W_{12} \subseteq {\alpha_1}^* TM$ be the set:*

$$\{(t, v) \in {\alpha_1}^* TM \colon (\alpha_2(t), \eta_1(v)) \in V_2\}.$$

Then W_{12} is open and ${\Psi_1}^{-1}(U_{12}) = \Gamma^x_{S^1}(W_{12})$.

Here $\Gamma^x_{S^1}(W_{12})$ is the set of sections of ${\alpha_1}^* TM$ which take values in W_{12}.

Proof. The set W_{12} is open as it is the preimage of an open set via a continuous map. To show the second statement we need to prove that $\gamma \in \Gamma^x_{S^1}({\alpha_1}^* TM)$ takes values in W_{12} if and only if $\Psi_1(\gamma) \in U_2$ (by construction we already have $\Psi_1(\gamma) \in U_1$).

So let $\gamma \in \Gamma^x_{S^1}(\alpha^* TM)$ and let $\tilde{\gamma}$ be the image of γ in $L^x TU$. Thus $\gamma(t) = (t, \tilde{\gamma}(t))$. Now γ takes values in W_{12} if and only if:

$$\Big(\alpha_2(t), \eta(\tilde{\gamma}(t))\Big) \in V_2$$

for all $t \in S^1$. That is to say, if and only if $(\alpha_2, \eta_*(\tilde{\gamma})) \in L^x V_2$. By definition, this is equivalent to the statement that $\eta_*(\tilde{\gamma}) \in U_{\alpha_2}$. Now $\eta_*(\tilde{\gamma}) = \Psi_1(\gamma)$ so γ takes values in W_{12} if and only if $\Psi_1(\gamma) \in U_{\alpha_1} \cap U_{\alpha_2}$. □

Proposition 3.7 *The transition function:*

$$\Phi_{12} \coloneqq {\Psi_1}^{-1}\Psi_2 \colon {\Psi_1}^{-1}(U_{12}) \to {\Psi_2}^{-1}(U_{12})$$

is a diffeomorphism.

Proof. We define $W_{21} \subseteq {\alpha_2}^* TM$ as the set of $(t, v) \in \alpha_2^* TM$ with $(\alpha_1(t), \eta_1(v)) \in V_1$. As for W_{12}, $\Psi_2^{-1}(U_{12}) = \Gamma^x_{S^1}(W_{21})$.

The idea of the proof is to set up a diffeomorphism between W_{12} and W_{21}. Our assumptions on C^x–maps say that the resulting map on sections is a diffeomorphism. Finally, we show that this diffeomorphism is the transition function defined in the statement of this proposition.

Let $\theta_1 \colon W_{12} \to TM$ be the map:

$$\theta_1(t, v) = (\pi \times \eta_2)^{-1}(\alpha_2(t), \eta_1(v)).$$

The definition of W_{12} ensures that $(\alpha_2(t), \eta_1(v)) \in V_2$ for $(t, v) \in W_{12}$ and this is the image of $\pi \times \eta_2$. Hence θ_1 is well-defined. Define $\theta_2 \colon W_{21} \to TM$ similarly. These are both smooth maps.

Notice that $\pi(\pi \times \eta_i)^{-1} \colon V_i \subseteq U_i \times U_i \to U_i$ is the projection onto the first factor and $\eta_i(\pi \times \eta_i)^{-1} \colon V_i \to U_i$ is the projection onto the second. Thus $\pi\theta_1(t, v) = \alpha_2(t)$. Hence $\theta_1 \colon W_{12} \to TM$ is such that $(t, \theta_1(t, v)) \in {\alpha_2}^* TM$ for all $(t, v) \in W_{12}$. Then:

$$\Big(\alpha_1(t), \eta_2(\theta_1(t, v))\Big) = (\alpha_1(t), \eta_1(v)) \in V_1$$

so $(t, \theta_1(t, v)) \in W_{21}$. Hence we have a map $\phi_{12} \colon W_{12} \to W_{21}$ given by:

$$\phi_{12}(t, v) = (t, \theta_1(t, v)).$$

Similarly we have a map $\phi_{21} \colon W_{21} \to W_{12}$. These are both smooth since the composition with the inclusion into $S^1 \times TM$ is smooth.

Consider the composition $\phi_{21}\phi_{12}(t,v)$. Expanding this out yields:

$$\begin{aligned}\phi_{21}\phi_{12}(t,v) &= \phi_{21}(t,\theta_1(t,v))\\ &= (t,\theta_2(t,\theta_1(t,v)))\\ &= \Big(t,(\pi\times\eta_1)^{-1}(\alpha_1(t),\eta_2(\theta_1(t,v)))\Big)\\ &= \Big(t,(\pi\times\eta_1)^{-1}(\alpha_1(t),\eta_1(v))\Big)\\ &= \Big(t,(\pi\times\eta)^{-1}(\pi(v),\eta_1(v))\Big)\\ &= (t,v).\end{aligned}$$

The penultimate line is because $(t,v)\in {\alpha_1}^*TM$ so $\pi(v)=\alpha_1(t)$.

Hence ϕ_{21} is the inverse of ϕ_{12} and so $\phi_{12}\colon W_{12}\to W_{21}$ is a diffeomorphism. Thus by lemma 2.8, the map ${\phi_{12}}_*$ is a diffeomorphism from ${\Psi_1}^{-1}(U_{12})$ to ${\Psi_2}^{-1}(U_{12})$. We just need to show that this is the transition function. It is sufficient to show that $\Psi_2{\phi_{12}}_* = \Psi_2\Phi_{12}$. The right-hand side is, by definition, Ψ_1, which satisfies:

$$\Psi_1(\gamma)(t) = \eta_1(\tilde{\gamma}(t))$$

where $\tilde{\gamma}\colon S^1\to TM$ is such that $\gamma(t)=(t,\tilde{\gamma}(t))$. On the other side:

$$\begin{aligned}{\phi_{12}}_*(\gamma)(t) &= \phi_{12}(\gamma(t))\\ &= \Big(t,\theta_1(t,\tilde{\gamma}(t))\Big)\\ &= \Big(t,(\pi\times\eta_2)^{-1}(\alpha_2(t),\eta_1(\tilde{\gamma}(t)))\Big),\end{aligned}$$

hence:

$$\begin{aligned}\Psi_2{\phi_{12}}_*(\gamma)(t) &= \eta_2(\pi\times\eta_2)^{-1}(\alpha_2(t),\eta_1(\tilde{\gamma}(t)))\\ &= \eta_1(\tilde{\gamma}(t)),\end{aligned}$$

as required. Thus ${\phi_{12}}_* = \Phi_{12}$ and so the transition functions are diffeomorphisms. □

We therefore have a smooth atlas for L^xM.

4 Topology

Following [KM97, ch 27] we proceed to topologise L^xM with the inductive topology for the chart maps. Our concern now is to determine some topological properties of L^xM. The key is theorem B. Once we have proved this then the passage from $L^x\mathbb{R}$ to L^xM is straightforward. We also show that the inclusion $LM\to L^xM$ is a homotopy equivalence.

4.1 Submanifolds

Proposition 4.1 *Let M, N be finite dimensional smooth manifolds with M an embedded submanifold of N. Then L^xM is an embedded submanifold of L^xN.*

Proof. By the tubular neighbourhood theorem there is an open neighbourhood U of M in N, a smooth vector bundle $E \to M$, and a diffeomorphism $\phi\colon E \to U$ which maps the zero section to the inclusion of M in U. Let $\eta\colon TM \to M$ be a local addition over M with neighbourhood $V \subseteq M \times M$. Let $E_V \subseteq E \times E$ be the restriction of $E \times E$ to V. Choose a connection on E. Let $(u, v) \in V$. Let $p = (\pi \times \eta)^{-1}(u, v)$. This lies in T_uM so the path $t \mapsto tp$ goes from the zero vector in T_uM to p. Applying η results in a path from $\eta(0_u) = u$ to $\eta(p) = v$. Let $P(u, v)\colon E_u \to E_v$ be the operator defined by parallel transport along this path.

Using the connection a point in TE can be thought of as a quadruple (u, p, v, w) where $u \in M, p \in T_uM$, and $v, w \in E_u$ with the projection $TE \to E$ being $(u, p, v, w) \to (u, v)$ (we include u in the notation to emphasise the fibre). Define $\eta^E\colon TE \to E$ by $\eta^E(u, p, v, w) = (\eta(p), P(u, \eta(p))(v + w))$. Then $\pi^E \times \eta^E\colon TE \to E \times E$ is:

$$(\pi^E \times \eta^E)(u, p, v, w) = \Big((u, v), (\eta(p), P(u, \eta(p))(v + w))\Big).$$

The projection of this to $M \times M$ is $(u, \eta(p))$; so the image of $\pi^E \times \eta^E$ is in E_V. Since the map $(u, p) \to (u, \eta(p))$ is onto V, varying v and w shows that we can get all of E_V. The inverse map is:

$$\Big((u, v), (x, w)\Big) \mapsto ((\pi \times \eta)^{-1}(u, x), v, P(\pi \times \eta^{-1}(u, x))(w) - v).$$

This is smooth, so $\pi^E \times \eta^E$ is a diffeomorphism onto E_V and thus defines a local addition over the whole of E.

Using the diffeomorphism $E \cong U$ we transfer this to U and so get a local addition for $M \subseteq N$. The charts that this defines make up part of the smooth atlas for N. Taking a chart based at a smooth loop α in M, we see that the inclusion of L^xM in L^xN looks like the inclusion of $\Gamma^x_{S^1}(\alpha^*TM)$ in $\Gamma^x_{S^1}(\alpha^*TN)$. This, in turn, looks like the inclusion of $L^x\mathbb{R}^k$ in $L^x\mathbb{R}^n$. Hence L^xM is an embedded submanifold of L^xN. □

Corollary 4.2 *Let M be a closed finite dimensional smooth manifold. Then the following properties hold for L^xM if they hold for $L^x\mathbb{R}^n$: separable, metrisable, Lindelöf, paracompact, normal, smoothly regular, smoothly paracompact, and smoothly normal.*

Proof. There is an embedding of M as a submanifold of $\mathbb{R}^n$. As it is compact, the image is closed in $\mathbb{R}^n$. We therefore have an embedding of L^xM in $L^x\mathbb{R}^n$ which is also closed. The properties listed are all inherited by closed subspaces. □

4.2 Homotopy Equivalence

One remarkable fact about the spaces LM and L^0M is that they are homotopy equivalent. The standard method of this is to find *mollifiers* which "smooth out" continuous functions. This approach does not work with an arbitrary family of maps. The paths defined by the homotopy lie in the space of smooth loops at all points except one end-point, therefore if smooth loops are not dense in the given family of maps this homotopy cannot be continuous. However using the fact that L^xM is a smooth manifold one can still show that the inclusion $LM \to L^xM$ is a homotopy equivalence.

The first step is to define the reverse map. The basic idea is to use a mollifier but we have to be a bit selective. Let M be a closed smooth finite dimensional manifold. Via an embedding, regard M as a submanifold of some Euclidean

space, $\mathbb{R}^n$. By the tubular neighbourhood theorem, there is a neighbourhood of M in $\mathbb{R}^n$ which retracts onto M. That is, there is some open neighbourhood $U \subseteq \mathbb{R}^n$ of M and a map $p\colon U \to M$ which is the identity on M. Let $\eta\colon TM \to M \times M$ be a local addition on M with image V.

As M is compact we can find $\mu > 0$ such that if $x, y \in M$ are such that $\|x - y\| < \mu$ then $(x, y) \in V$. We can also find $\epsilon > 0$ such that if $x \in M$ and $y \in \mathbb{R}^n$ are such that $\|x - y\| < \epsilon$ then $y \in U$ and $\|x - p(y)\| < \mu$.

Lemma 4.3 *There is a continuous function $\delta\colon L^0\mathbb{R}^n \to \mathbb{R}$ such that for $\gamma \in L^0\mathbb{R}^n$ then whenever $|s - t| < \delta(\gamma)$, $\|\gamma(s) - \gamma(t)\| < \epsilon$.*

Proof. Let $\gamma \in L^0\mathbb{R}^n$. Then there is some $\delta_\gamma > 0$ such that whenever $|s - t| < \delta_\gamma$, $\|\gamma(s) - \gamma(t)\| < \epsilon/3$. Let β be such that $\|\beta - \gamma\|_\infty < \epsilon/3$. Then whenever $|s - t| < \delta_\gamma$,

$$\begin{aligned}\|\beta(s) - \beta(t)\| &\leq \|\beta(s) - \gamma(s)\| + \|\gamma(s) - \gamma(t)\| + \|\beta(t) - \gamma(t)\| \\ &\leq 2\|\beta - \gamma\|_\infty + \|\gamma(s) - \gamma(t)\| \\ &< \epsilon.\end{aligned}$$

Now $L^0\mathbb{R}^n$ is metrisable, hence paracompact and Hausdorff. It therefore admits partitions of unity. Choose a partition, $\{\tau_\lambda\colon \lambda \in \Lambda\}$, subordinate to the cover of open balls of radius $\epsilon/3$. For each $\lambda \in \Lambda$ choose γ_λ such that the support of τ_λ is within the $\epsilon/3$–ball around γ_λ. Let $\delta_\lambda := \delta_{\gamma_\lambda}$. Define $\delta\colon L^0\mathbb{R}^n \to \mathbb{R}$ by:

$$\delta(\gamma) = \sum_{\lambda \in \Lambda} \delta_\lambda \tau_\lambda(\gamma).$$

For $\gamma \in L^0\mathbb{R}^n$, consider the set $\Lambda(\gamma) := \{\lambda : \tau_\lambda(\gamma) \neq 0\}$. This set is finite and if $\lambda \in \Lambda(\gamma)$ then $\|\gamma - \gamma_\lambda\| < \epsilon/3$. As $\delta(\gamma)$ is a convex sum of the set $\{\delta_\lambda : \lambda \in \Lambda(\gamma)\}$, there is some $\lambda \in \Lambda(\gamma)$ with $\delta(\gamma) \leq \delta_\lambda$. Whereupon we have that if $|s - t| < \delta(\gamma)$, $\|\gamma(s) - \gamma(t)\| < \epsilon$ as required. □

Using this, we define a continuous map $R\colon L^0\mathbb{R}^n \to L\mathbb{R}^n$ with the property that $\|\gamma - R(\gamma)\|_\infty < \epsilon$ for all γ. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth bump function with support in $[-1, 1]$ and $\int_{\mathbb{R}} \phi = 1$. For $r > 0$ let $\phi_r\colon \mathbb{R} \to \mathbb{R}$ be the function $t \mapsto r\phi(t/r)$. This has support in $[-r, r]$ and satisfies $\int_{\mathbb{R}} \phi_r = 1$. Define $R\colon L^0\mathbb{R}^n \to L\mathbb{R}^n$ by:

$$\gamma \mapsto \gamma * \phi_{\delta(\gamma)}.$$

By this we mean that each component of γ is regarded as a map with domain $\mathbb{R}$ and is convoluted with the map $\phi_{\delta(\gamma)}$.

Lemma 4.4 *The map $R\colon L^0\mathbb{R}^n \to L\mathbb{R}^n$ is continuous and satisfies $\|\gamma - R(\gamma)\|_\infty < \epsilon$ for all γ.*

Proof. Let $\gamma \in L^0\mathbb{R}$ and $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$. Then:

$$(\gamma * \beta)(t) = \int_{\mathbb{R}} \gamma(s)\beta(t - s)ds.$$

Hence:

$$\begin{aligned}(\gamma * \beta)(t+1) &= \int_{\mathbb{R}} \gamma(s)\beta(t+1-s)ds \\ &= \int_{\mathbb{R}} \gamma(\tilde{s}+1)\beta(t-\tilde{s})d\tilde{s} \\ &= \int_{\mathbb{R}} \gamma(\tilde{s})\beta(t-\tilde{s})d\tilde{s} \\ &= (\gamma * \beta)(t),\end{aligned}$$

whence $R(\gamma)$ is periodic so can be viewed as a map with domain S^1. The convolution of a continuous function by a smooth function is again smooth so $\gamma * \beta \in L\mathbb{R}$, with derivative $D(\gamma * \beta) = \gamma * D\beta$. Hence the image of R is $L\mathbb{R}^n$ as required.

To show that it is continuous, it is sufficient to show that the map $L^0\mathbb{R} \times (0,\infty) \to L\mathbb{R}$, $(\gamma, r) \mapsto \gamma * \phi_r$ is continuous. Now for bounded maps on $\mathbb{R}$, the map $(\alpha, \beta) \mapsto \alpha * \beta$ is bilinear and satisfies:

$$\left\|\alpha * \beta\right\|_\infty \leq \|\alpha\|_\infty \left\|\beta\right\|_\infty$$

where, by abuse of notation, we have used $\|\cdot\|_\infty$ for the supremum norm for bounded functions on $\mathbb{R}$. Hence for $\alpha, \beta \in L^0\mathbb{R}$, $r, s \in (0,\infty)$, and $k \in \mathbb{N}$,

$$\left\|\alpha * (\phi_r)^{(k)} - \beta * (\phi_s)^{(k)}\right\|_\infty \leq \|\alpha\|_\infty \left\|(\phi_r)^{(k)} - (\phi_s)^{(k)}\right\|_\infty + \left\|\alpha - \beta\right\|_\infty \left\|(\phi_s)^{(k)}\right\|_\infty .$$

This shows that providing α and β are close and providing $(\phi_r)^{(k)}$ and $(\phi_s)^{(k)}$ are close then $(\alpha * \phi_r)^{(k)}$ and $(\beta * \phi_r)^{(k)}$ are close. Thus to show that R is continuous it is sufficient to observe that the map $r \to \phi_r$ is continuous as a path into $C^\infty(\mathbb{R}, \mathbb{R})$ and this is straightforward.

Finally, let $\gamma \in L^0\mathbb{R}^n$. For $t \in S^1$, as $\int_{\mathbb{R}} \phi_r = 1$, $\gamma(t) - R(\gamma)(t)$ is given by:

$$\gamma(t) - \int_{\mathbb{R}} \gamma(s)\phi_r(t-s)ds = \int_{\mathbb{R}} (\gamma(t) - \gamma(s))\phi_{\delta(\gamma)}(t-s)ds.$$

Now $\phi_{\delta(\gamma)}(t-s)$ is zero outside $[t-\delta(\gamma), t+\delta(\gamma)]$ and on this interval $\left|\gamma(t) - \gamma(s)\right| < \epsilon$, by definition of $\delta(\gamma)$. Hence $\left|\gamma(t) - R(\gamma)(t)\right| < \epsilon$ as required. □

Corollary 4.5 *There is a continuous map $R_M \colon L^0M \to LM$ with the property that for all $\gamma \in L^0M$, $(\gamma(t), R_M(\gamma)(t)) \in V$ for all $t \in S^1$.*

Proof. We restrict the map R to the domain L^0M. By construction, $R(\gamma)$ takes values in U, the neighbourhood of M. The map R_M is the composition of this with the projection $p \colon U \to M$. The required property holds because of the choices made. □

Theorem 4.6 *Let $L^x\mathbb{R}$ be a class of maps satisfying the assumptions of section 1. Then the inclusion $LM \to L^xM$ is a homotopy equivalence.*

Proof. The reverse map is the composition of the inclusion of L^xM in L^0M with the map R_M. We will denote this again by R_M.

By construction, for $\gamma \in L^x M$, $(R_M(\gamma), \gamma)$ takes values in V. Define $H\colon L^x M \times [0,1] \to L^x M$ by:

$$H(\gamma, s) = \pi_2 \eta(s\eta^{-1}(R_M(\gamma), \gamma)),$$

where in TM we have used the natural $\mathbb{R}$–action on the fibres and π_2 is the projection onto the second factor. This is continuous as it is the composition of continuous maps. For $s = 1$ we have $H(\gamma, 1) = \pi_2(R_M(\gamma), \gamma) = \gamma$. For $s = 0$, $H(\gamma, 0) = \pi_2(\eta(R_M(\gamma), R_M(\gamma))) = R_M(\gamma)$. Hence H is the required homotopy.

The same homotopy works for smooth maps, showing that the composition $LM \to L^x M \to LM$ is homotopic to the identity. □

4.3 Based Loops

All of our discussion so far holds for based loops as well as free loops. For the main part, the only modification needed is the insertion of the word "based" at appropriate points. The only place where more is required is in proving the homotopy equivalence. The problem there is that the result of applying a mollifier to a based continuous loop may no longer be based. The fix is simple, however, since at that point in the construction of the homotopy equivalence we are dealing with loops in $\mathbb{R}^n$. We can therefore define the based mollifier R_0 in terms of the map R defined in section 4.2 as

$$R_0(\gamma) = R(\gamma) - R(\gamma)(0)$$

(we are tacitly assuming that the basepoint of $\mathbb{R}^n$ is the origin). To ensure that the resulting map $R_0(\gamma)$ has the properties analogous to corollary 4.5 we need to replace ϵ by $\epsilon/2$ in section 4.2.

The relationship between based loops and free loops is an important one. In homotopy theory there is a fibration

$$\Omega^0 M \to L^0 M \to M$$

and so, via the homotopy equivalences, we can deduce that

$$\Omega^x M \to L^x M \to M$$

is also a fibration. This describtion, however, is not one of the flavour of differential topology. A more suitable description would be that it is a locally trivial fibre bundle. This will follow from the following theorem. Let $e_0\colon L^x M \to M$ be the evalutation map at 0, $\gamma \mapsto \gamma(0)$.

Theorem 4.7 *Let M, P be a smooth finite dimensional orientable manifolds without boundary. Suppose that P is an embedded submanifold of M with tubular neighbourhood $U \subseteq M$ and normal bundle $N \to P$. Define*

$$L^x_P M := \{\gamma \in L^x M : \gamma(0) \in P\}$$

and $L^x_U M$ similarly. Then $L^x_P M$ is an embedded submanifold of $L^x M$ with tubular neighbourhood $L^x_U M$ and normal bundle ${e_0}^* N$. Moreover, the evalutation map $e_0\colon L^x M \to M$ takes the quadruple $(L^x M, L^x_P M, L^x_U M, {e_0}^* N)$ to (M, P, U, N) preserving all the structure.

Proof. We omit the proof that $L^x_P M$ is a smooth submanifold of $L^x M$ as this is a simple modification of the work of previous sections; the model space is $L^x_{\mathbb{R}^k}\mathbb{R}^n$. The case of $L^x_U M$ is simpler as it is an open submanifold of $L^x M$.

Let $\pi\colon N \to P$ be the projection and let $\phi\colon U \to N$ be the diffeomorphism. Our strategy is to find a continuous map $\Psi\colon U \to \mathrm{Diff}_c(U)$, where $\mathrm{Diff}_c(U)$ is diffeomorphisms of U that are the identity outside a compact set, such that $\Psi_u(u) = \pi\phi(u)$. Using this, we define the diffeomorphism $L^x_U M \to e_0{}^* N$ by

$$\alpha \mapsto \left(\Psi_{\alpha(0)}(\alpha), \phi(\alpha(0))\right)$$

with inverse

$$(\beta, v) \mapsto \Psi^{-1}_{\phi^{-1}(v)}(\beta).$$

That these are smooth follows from the fact that they are defined entirely in terms of smooth maps of the original manifolds and these induce smooth maps on our loop spaces by assumption.

Thus we need to find the map $\Psi\colon U \to \mathrm{Diff}_c(U)$ with the appropriate conditions. We actually define the map for N as there we can use the vector bundle structure and then transfer it to U via the diffeomorphism ϕ.

The first stage of defining Ψ is to define a map $s\colon N \to \Gamma_c(N)$, the space of sections of N with compact support. Let $\{(U_\lambda, \nu_\lambda, \rho_\lambda) : \lambda \in \Lambda\}$ be a family of triples where

1. $\{U_\lambda : \lambda \in \Lambda\}$ is a locally finite open cover of P such that each U_λ has compact closure in P;
2. $\nu_\lambda\colon \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbb{R}^n$ is a trivialisation of N over U_λ;
3. $\{\rho_\lambda : \lambda \in \Lambda\}$ squares to a partition of unity subordinate to the cover $\{U_\lambda\}$; that is, $\rho_\lambda\colon P \to \mathbb{R}$ is a bump function with support in U_λ and $\sum_\lambda \rho_\lambda(x)^2 = 1$ for all $x \in P$.

Let $\tilde{\nu}_\lambda\colon \pi^{-1}(U_\lambda) \to \mathbb{R}^n$ be the composition of ν_λ with the projection onto $\mathbb{R}^n$. Define $s\colon N \to \Gamma_c(N)$ by

$$s(v)(x) \coloneqq \sum_\lambda \rho_\lambda(\pi(v))\rho_\lambda(x)\nu_\lambda^{-1}(x, \tilde{\nu}_\lambda(v)).$$

The sum is well-defined since $\rho_\lambda(\pi(v))\rho_\lambda(x)$ can only be non-zero if both $\pi(v)$ and x are in the domain of ν_λ. Each $s(v)$ is clearly a section and its support is contained in the finite union $\bigcup\{U_\lambda : \pi(v) \in U_\lambda\}$, and hence has compact support. Observe that

$$s(v)(\pi(v)) = \sum_\lambda \rho_\lambda(\pi(v))\rho_\lambda(\pi(v))\nu_\lambda^{-1}(\pi(v), \tilde{\nu}_\lambda(v)) = \sum_\lambda \rho_\lambda(\pi(v))^2 v = v.$$

There is a canonical embedding of N in TN as the vertical tangent bundle and thus we can extend any section σ of N to a vector field on N by defining $\tilde{\sigma}(v) = \sigma(\pi(v))$. If the original section had compact support then the resulting vector field will have compact horizontal support. We wish to apply this proceedure to the sections that we have defined above, but we also wish to ensure that the resulting vector fields have genuine compact support. To do that, we choose an inner product on the fibres of N which varies smoothly over P and a bump function $\tau\colon \mathbb{R} \to \mathbb{R}$ which takes the value 1 on $[0, 1]$ and is zero above, say, 2. Define

$$X_v(u) \coloneqq -\tau\left(\|u\|^2 / (1 + \|v\|^2)\right) s(v)(\pi(u)).$$

This has compact support, is a vertical vector field, and for $u \in N_v$ with $\|u\| \leq \|v\|$ then $X_v(u) = -v$.

We therefore have a continuous map $N \to \Xi_c(N)$. We compose this with the exponential map $\exp\colon \Xi_c(N) \to \mathrm{Diff}_c(N)$ to define $\Psi\colon N \to \mathrm{Diff}_c(N)$. The properties of X_v translate into properties of Ψ_v. As X_v is a vertical vector field, Ψ_v preserves the fibres of N. Most importantly, $\Psi_v(v) = 0_v$.

This is the required map and so establishes $L^x_U M$ as the tubular neighbourhood of $L^x_P M$. It is clear from the setup that the evaluation map has the properties stated in the theorem. □

Corollary 4.8 *The evaluation map $L^x M \to M$ is a locally trivial fibre bundle with fibre $\Omega^x M$.*

Proof. Take $P = \{x_0\}$ to be the basepoint and U the codomain of a chart near x_0 with domain $\mathbb{R}^n$. □

5 Circle Actions

The diffeomorphism group of the circle acts on maps with domain S^1 by precomposition. It is usual to assume that $L^x\mathbb{R}$ is closed under this action, whence we get an action on $L^x M$ for M a smooth finite dimensional manifold. We would like to transfer knowledge of that action from $L^x\mathbb{R}$ to $L^x M$.

5.1 Transferring the Action

We are going to prove an inheritance result which states that the action on $L^x M$ is the same as that on $L^x\mathbb{R}^n$. This will be an easy corollary of theorem B. The more important point of this section is to consider what types of action there are.

Definition 5.1 *Let M be a smooth finite dimensional manifold. Let $G \subseteq \mathrm{Diff}(S^1)$ be a sub–Lie group of the set of diffeomorphisms of the circle. We define the following possible types of action of G on $L^x M$.*

1. *The action is by bijections.*
2. *The action is by homeomorphisms.*
3. *The action is by diffeomorphisms.*
4. *The action map, $G \times L^x M \to L^x M$, is continuous.*
5. *The action map, $G \times L^x M \to L^x M$, is smooth.*
6. *The representation map, $G \to \mathrm{Homeo}(L^x M)$, is continuous.*
7. *The representation map, $G \to \mathrm{Diff}(L^x M)$, is smooth.*

The diffeomorphism group, $\mathrm{Diff}(S^1)$, is an open subset of LS^1 and thus inherits the structure of a smooth manifold. The circle, acting by rigid rotations, is a subset of $\mathrm{Diff}(S^1)$ and the inherited structure is the same as its usual one.

Although these levels have been written in a necessarily linear form, the relationships between them are more complicated than this suggests. For example,

a continuous representation map does not necessarily imply a continuous action map as the evaluation map $\mathrm{Homeo}(X) \times X \to X$ is not necessarily jointly continuous.

Proposition 5.2 *All the levels defined are inherited by $L^x M$ from $L^x\mathbb{R}$.*

Proof. As diffeomorphisms of the circle act linearly on $L^x\mathbb{R}$ this proposition holds for $M = \mathbb{R}^n$ simply by taking finite products. The result for general M follows from theorem B. Since $L^x M$ is an embedded submanifold of $L^x\mathbb{R}^n$, a map into $L^x M$ is continuous or smooth if and only if it is continuous or smooth into $L^x\mathbb{R}^n$; and the restriction to $L^x M$ of a continuous or smooth map from $L^x\mathbb{R}^n$ is again continuous or smooth.

For the representation maps, we are being deliberately vague about the topologies on the homeomorphism and diffeomorphism groups. There is considerable freedom in choosing this topology and we wish to allow for this freedom, only assuming that the topologies are compatbile for M as for $\mathbb{R}^n$. Then as $L^x M$ is a smooth retract of an open subset of $L^x\mathbb{R}^n$, the restriction map from the subspace of homeomorphisms, resp. diffeomorphisms, of $L^x\mathbb{R}^n$ which preserve $L^x M$ as a set to the set of homeomorphisms, resp. diffeomorphisms, of $L^x M$ is defined and continuous, resp. smooth. As the representation map of G factors through this map its properties are inherited from those of the representation on $L^x\mathbb{R}$. □

5.2 Continuity of Linear Circle Actions

The most common action considered on loop spaces is that of the circle itself. In light of the inheritance properties of circle actions, it seems a good idea to consider the general case of the circle acting on a locally convex topological vector space. As this is, by its very nature, more in the realm of functional analysis than differential topology, at each stage we shall consider how it applies to the examples of smooth loops and continuous loops in order to ground the discussion in terms familiar to the differential topologist.

We start with a more detailed discussion of what it might mean for a circle action to be "continuous". There are several "levels" of continuity that one could consider, more than those listed in the previous section. The following definition contains the ones that we think are interesting or useful.

Definition 5.3 *Let E be a lctvs. Suppose that the circle acts on E by linear maps, not necessarily continuous. Let R_t be the linear map corresponding to $t \in S^1$. We define the following levels of continuity for this action:*

1. *The representation is continuous; that is, the action induces a continuous map $S^1 \to \mathcal{L}_b(E)$. Here, $\mathcal{L}_b(E)$ denotes the space of continuous linear maps from E to itself equipped with the* strong *topology; that is, the topology of uniform convergence on bounded sets.*

2. *The action is continuous; that is, the action is continuous as a map $S^1 \times E \to E$.*

3. *The action is separately continuous; that is, for each $t \in S^1$ then $x \to R_t x$ is a continuous map $E \to E$, and for each $x \in E$ then $t \to R_t x$ is a continuous map $S^1 \to E$.*

4. *The action is by equicontinuous linear maps; that is to say, for each* 0*–neighbourhood* V *in* E *there is a* 0*–neighbourhood* U *such that* $R_t U \subseteq V$ *for all* $t \in S^1$.

5. *There is a* 0*–basis of* S^1*–invariant sets.*

6. *The action is by continuous linear maps; that is, each* R_t *is continuous.*

7. *The topology on* E *is* S^1*–invariant.*

The strong topology is the finest topology that one would sanely use. Thus positive results for the strong topology will propagate forwards to any coarser topology. A 0–basis for this topology consists of the sets:

$$N(B, U) := \{T \in \mathcal{L}(E) : T(B) \subseteq U\}$$

where B, U are subsets of E with B bounded and U a 0–neighbourhood. If E is a Banach space then this is the usual topology which is normable with norm $\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\}$..

We shall now show how the list in definition 5.3 is, roughly, from the strictest to the weakest. We start with just the results that apply to all lctvs.

Proposition 5.4 *Let* E *be a lctvs with an action of the circle by linear maps. We have the following links between the levels of continuity:*

(i). 4 is equivalent to 5;

(ii). 6 is equivalent to 7;

(iii). 2 is equivalent to having both 3 and 4;

(iv). 3 implies 6;

(v). 1 implies 3.

Before proving this we remark that the reason why 1 does not automatically imply 2 is because the evaluation map $\mathcal{L}_b(E) \times E \to E$ is not, in general, continuous but only separately continuous. Thus the action map is separately continuous as it factors as:

$$S^1 \times E \to \mathcal{L}_b(E) \times E \to E$$

but we cannot deduce from this that it is continuous.

Proof. The equivalences (i) and (ii) are obvious, as is the implication (iv). The deduction of 3 from 2 is also obvious. We have already explained (v).

Thus only (iii) remains and of that we need to show that 2 implies 4 and that together 3 and 4 imply 2.

To show that 2 implies 4 let V be an open 0–neighbourhood in E. By assumption, for each $t \in S^1$ there is some open 0–neighbourhood U_t and $\delta_t > 0$ such that $(t - \delta_t, t + \delta_t) \times U_t$ maps into V. As S^1 is compact there is some finite set $\{t_1, \ldots, t_n\}$ such that the intervals $\{(t_j - \delta_{t_j}, t_j + \delta_{t_j})\}$ cover S^1. Let $U = \bigcap_{j=1}^n U_{t_j}$. Then U is a finite intersection of open 0–neighbourhoods, hence is one itself. For $t \in S^1$ there is some j such that $t \in (t_j - \delta_{t_j}, t_j + \delta_{t_j})$ whence, as $U \subseteq U_{t_j}$, $R_t(U) \subseteq V$. Thus the action is by equicontinuous linear maps.

For the converse we assume both 3 and 4. Let $x \in E$ and $t \in S^1$. Let V be a convex 0–neighbourhood which, by 4, we may assume to be S^1–invariant. Then $\frac{1}{2}V$ is also a convex, S^1–invariant 0–neighbourhood so as the map $s \to R_s x$ is continuous at t there is some $\delta > 0$ such that if $|s| < \delta$ then $R_t x - R_{t+s} x \in \frac{1}{2}V$. Let $s \in S^1$ be such that $|s| < \delta$ and let $y \in x + \frac{1}{2}V$. We have:

$$\begin{aligned} R_t x - R_{t+s} y &= R_t x - R_{t+s} x + R_{t+s} x - R_{t+s} y \\ &= R_t x - R_{t+s} x + R_{t+s}(x - y) \\ &= \frac{1}{2}(2R_t x - 2R_{t+s} x) + \frac{1}{2} R_{t+s}(2x - 2y). \end{aligned}$$

Now $2R_t x - 2R_{t+s} x$ and $2x - 2y$ both lie in V. As V is S^1–invariant, $R_{t+s}(2x - 2y)$ also lies in V. Thus as V is convex, $R_t x - R_{t+s} y$ is in V. Hence $(t-\delta, t+\delta) \times x + \frac{1}{2}V$ lies in the preimage of $R_t x + V$. Hence the action is continuous. □

There are more connections between these conditions if the space E has more structure.

As mentioned above the failure of 1 to automatically imply 2 is due to possibility that the evaluation map is not continuous. It is continuous if, and only if, E is normable. Thus we deduce:

Lemma 5.5 *Let E be a normable lctvs with an action of the circle by linear maps. Then 1 implies 2.* □

A more general class of spaces that allows us to strengthen the links is the family of *barrelled* lctvs. This is a technical property of lctvs which we shall not describe here, we merely need one of its well-known consequences. It follows from [Jar81, 11.1.5] and Baire's theorem that $L^0\mathbb{R}$ and $L\mathbb{R}$ are barrelled.

Proposition 5.6 *Let E be a barrelled lctvs with an action of the circle by linear maps. Then 3 implies 4. Hence each of 1 and 3 imply 2.*

Proof. The proof that 3 implies 4 is similar to [Sch71, III§5.3]. That the action is separately continuous means that the map $S^1 \to \mathcal{L}(E)$ is well-defined and is continuous for the topology of uniform convergence on all *finite* sets. Thus the image of S^1 in $\mathcal{L}(E)$ is simply bounded and hence, as E is barrelled, equicontinuous.

Since 3 and 4 together imply 2 we therefore have that 3 alone implies 2. Also as 1 implies 3 we also have that 1 implies 2. □

A useful property of $L\mathbb{R}$ is that closed bounded subsets are compact; this follows from [Sch71, II§7.2] as it is a complete nuclear space.

Proposition 5.7 *Let E be a lctvs with an action of the circle by linear maps. Suppose that every closed, bounded subset of E is compact. Then 2 implies 1.*

Proof. We shall show that if the action is continuous then the map $S^1 \to \mathcal{L}(E)$ is continuous for the topology of uniform convergence on compact sets. The assumption on E then says that this is precisely the topology of uniform convergence on bounded sets.

So assume that the circle action on E is continuous. Let $C, V \subseteq E$ be such that C is compact and V is a convex, circled 0–neighbourhood. Let $t_0 \in S^1$. As the circle action is continuous then for each $c \in C$ there is some $\delta_c > 0$ and U_c a neighbourhood of c in E such that if $x \in U_c$ and $|t| < \delta_c$ then $R_{t_0+t} x - R_t c \in \frac{1}{2}V$.

The neighbourhoods $\{U_c\}$ cover C so there is some finite subset which will do; say, $U_1, \ldots, U_n$ corresponding to points $c_1, \ldots, c_n \in C$. Let δ be the minimum of the corresponding subfamily of $\{\delta_c\}$; then $\delta > 0$.

Let t be such that $|t| < \delta$. Let $c \in C$, then there is some j such that $c \in U_j$. Thus $R_{t_O+t}c - R_{t_0}c_j \in \frac{1}{2}V$. Now the choice of c_j depended only on c and not on t. Therefore we also have $R_{t_O+0}c - R_{t_0}c_j \in \frac{1}{2}V$. Thus:

$$R_{t_0+t}c - R_{t_0}c = R_{t_0+t}c - R_{t_0}c_j + R_{t_0}c_j - R_{t_0}c$$

which, for the usual convexity reasons, lies in V. Hence for $|t| < \delta$, $R_{t_0+t} - R_{t_0}$ maps C into V. Thus the map $S^1 \to \mathcal{L}(E)$ is continuous for the topology of uniform convergence on compact subsets. □

5.3 Circle Actions on Loop Spaces

In this section we shall use the results of the previous one to determine how continuous are the circle actions on our example spaces. For convenience we list the technical properties of our spaces so that we know which of the above results apply. We also, for quick reference, list a 0–basis.

1. $L^0\mathbb{R}$ is barrelled. The topology is determined by the sets:

$$U(\epsilon) \coloneqq \{\gamma : \sup\{\left|\gamma(t)\right| : t \in S^1\} < \epsilon\}.$$

2. $L\mathbb{R}$ is barrelled and every closed bounded subset is compact. The topology is determined by the sets:

$$U(n, \epsilon) \coloneqq \{\gamma : \sup\{\left|\gamma^{(k)}(t)\right| : t \in S^1, 0 \leq k \leq n\} < \epsilon\}$$

We shall now determine how continuous is the action of rotation of loops on each of these spaces.

Proposition 5.8 *For both spaces the action is by continuous linear maps.*

Proof. We just need to show that the topology is S^1–invariant. It is sufficient to show this for the 0–neighbourhoods listed above. We have:

$$R_s U(\epsilon) = U(\epsilon),$$
$$R_s U(n, \epsilon) = U(n, \epsilon),$$

Thus in each case the topology is S^1–invariant and so the action is by continuous linear maps. □

Proposition 5.9 *For both spaces the action is by equicontinuous linear maps.*

Proof. From the previous proof it is obvious that the given 0–basis is of S^1–invariant sets. Hence for these three the action is by equicontinuous linear maps. □

Proposition 5.10 *The circle action on both of $L^0\mathbb{R}$ and $L\mathbb{R}$ is separately continuous.*

Proof. We already know that the circle acts by continuous linear maps which is half of separate continuity. Thus we need to show that for each loop γ then the map $t \to R_t\gamma$ is continuous.

We shall give the proof in full for $L\mathbb{R}$. The proof for $L^0\mathbb{R}$ is a simplification of this. We need to show that for $\gamma \in L\mathbb{R}$, $t_0 \in S^1$, and a 0–neighbourhood V then there is some $\delta > 0$ such that if $|t| < \delta$ then $R_{t_0+t}\gamma - R_{t_0}\gamma \in V$. It is sufficient to do this for $V = U(n, \epsilon)$ whence we need to show that $\left\|(R_{t_0+t}\gamma)^{(k)} - (R_{t_0}\gamma)^{(k)}\right\|_\infty < \epsilon$ for $0 \leq k \leq n$.

Expanding out the definition of the norm and using $(R_s\alpha)^{(k)}(t) = \alpha^{(k)}(t+s)$ we see that we want to ensure that:

$$\sup\{\left|\gamma^{(k)}(s+t) - \gamma^{(k)}(s)\right| : s \in S^1, 0 \leq k \leq n\} < \epsilon$$

whenever $|t| < \delta$. That such a $\delta > 0$ exists comes from the fact that the loops $\gamma, \gamma^{(1)}, \ldots, \gamma^{(n)}$ are all uniformly continuous on S^1 and there is only a finite number of them.

For $L^0\mathbb{R}$ the situation is slightly simplified in that we only need to consider γ and not any of its derivatives (which it may not have, of course). □

From propositions 5.6 and 5.7 we deduce the following.

Corollary 5.11 *The circle actions on $L^0\mathbb{R}$ and $L\mathbb{R}$ are continuous. The representation for the action on $L\mathbb{R}$ is also continuous.* □

Thus the action on $L\mathbb{R}$ is the best it can be. This is not true of $L^0\mathbb{R}$. We deduce this from a more general result which says that this is not the fault of the type of loop but rather of using a normed vector space of loops. Recall that a trigonometric polynomial is a (finite) linear span of sines and cosines.

Proposition 5.12 *Let $E \subseteq \mathrm{Map}(S^1, \mathbb{R})$ be an S^1–invariant vector space of loops which contains the trigonometric polynomials. Let p be an S^1–invariant semi-norm on E which restricts to a norm on the subspace of trigonometric polynomials. Let $(\tilde{E}, \|\cdot\|)$ be the associated Banach space. Then circle action is by equicontinuous linear maps but the associated representation is not continuous.*

Thus in this general case the only question to answer is whether or not the circle action itself is continuous.

Proof. As the set-up is S^1–invariant, the unit ball in $\tilde{E}$ is S^1–invariant and so the circle acts by equicontinuous linear maps.

Let $\delta > 0$. Choose $n \in \mathbb{N}$ such that $1/n < \delta$. As E contains the trigonometric polynomials it contains the loop $\gamma(t) = \cos(2\pi nt)v$ where $v \in \mathbb{R}$ is non-zero. By assumption on the semi-norm, γ represents a non-zero element in $\tilde{E}$. Let $h = 1/(2n)$, then $R_h\gamma = -\gamma$. Hence $\left\|(I - R_h)\gamma\right\| = 2\left\|\gamma\right\|$ and so $\|I - R_h\| \geq 2$. Thus the map $t \to R_t$ is not continuous into $\mathcal{L}_b(\tilde{E})$. □

In summary, the circle action on $L\mathbb{R}$ is as good as it can be whereas that on $L^0\mathbb{R}$ is almost that good and is as good as it can be given that it is a normed vector spaces.

5.4 Smooth Actions

We conclude with a comment on how smooth are the circle actions on $L\mathbb{R}$ and on $L^0\mathbb{R}$. As with continuity we can ask for different levels of smoothness.

For the positive results in this section we have to decide on a type of calculus. We choose the convenient calculus of [KM97]. This states that a map into a locally complete lctvs is smooth if and only if its composition with each continuous linear functional is a smooth map into $\mathbb{R}$. This provides us with test functions to determine whether or not a map is smooth. For the negative results we do not need to pick a calculus as for any calculus, continuous linear maps are certainly smooth and so we can still use them as test functions to determine if a map is not smooth.

We start with some positive results about $L\mathbb{R}$.

Proposition 5.13 *The action map $\rho\colon S^1 \times L\mathbb{R} \to L\mathbb{R}$ is smooth.*

Proof. As $L\mathbb{R}$ is a closed subspace of $C^\infty(\mathbb{R},\mathbb{R})$ and $\mathbb{R}$ is a covering space of S^1, it is clearly sufficient to show that the map $\tilde{\rho}\colon \mathbb{R} \times C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty(\mathbb{R},\mathbb{R})$, $(t,\zeta) \to (s \mapsto \zeta(s+t))$, is smooth. We need to show that it takes smooth curves in $\mathbb{R} \times C^\infty(\mathbb{R},\mathbb{R})$ to smooth curves in $C^\infty(\mathbb{R},\mathbb{R})$.

Let $c\colon \mathbb{R} \to \mathbb{R} \times C^\infty(\mathbb{R},\mathbb{R})$ be smooth. Let $\tilde{c} = \tilde{\rho} \circ c$. We can write $c = (c_1, c_2)$ for smooth curves $c_1\colon \mathbb{R} \to \mathbb{R}$ and $c_2\colon \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ since the obvious projection maps are smooth. Then for $t \in \mathbb{R}$

$$\tilde{c}(t) = (s \mapsto c_2(t)(s + c_1(t))).$$

By the exponential law, $\tilde{c}$ is smooth if and only if its adjoint, $\tilde{c}^\vee\colon \mathbb{R}^2 \to \mathbb{R}$ is smooth. This adjoint is

$$(s,t) \mapsto c_2(t)(s + c_1(t)).$$

Now as $c_2\colon \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ is smooth its adjoint, $c_2^\vee$, is also smooth, again by the exponential law. This is the map $(s,t) \mapsto c_2(t)(s)$. Thus $\tilde{c}^\vee$ is smooth as it factors as the composition

$$(s,t) \mapsto (s + c_1(t), t) \xrightarrow{c_2^\vee} c_2(t, s + c_1(t)).$$

Hence $\tilde{\rho}$ maps smooth curves to smooth curves and is thus smooth. □

Corollary 5.14 *The representation map $S^1 \to \mathcal{L}_b(L\mathbb{R})$ is smooth.*

Proof. This follows from the uniform boundedness principle, see [KM97, I.5.18]: a map into $\mathcal{L}_b(L\mathbb{R})$ is smooth if and only if all composites with evaluations at points in $L\mathbb{R}$ are smooth. □

Note that we cannot deduce from this that $S^1 \to \mathcal{L}_b(L\mathbb{R})$ is continuous since we have left the realm where the c^∞–topology agrees with the locally convex topology one.

Now we turn to the negative result and recall that here we do not assume a particular calculus.

Proposition 5.15 *Let $L^x\mathbb{R}$ be a class of loops satisfying the conditions 2, 3, and 5 of the introduction. Let $\gamma \in L^x\mathbb{R}$ be such that the map $S^1 \to L^x\mathbb{R}$, $t \mapsto R_t\gamma$, is smooth. Then $\gamma\colon S^1 \to \mathbb{R}$ is smooth.*

If, in addition, for all $\gamma \in L\mathbb{R}$ then the maps $S^1 \to L\mathbb{R}$, $t \mapsto R_t\gamma$, are smooth then the above becomes an if–and–only–if.

Proof. Let $e_0 \colon L^x\mathbb{R} \to \mathbb{R}$ be the evaluation map at 0. This is continuous by the assumptions and hence is smooth. As $S^1 \to L^x\mathbb{R}$, $t \mapsto R_t\gamma$, is smooth its composition with e_0 is a smooth map $S^1 \to \mathbb{R}$. This composition is $t \mapsto e_0(R_t\gamma) = \gamma(0 + t) = \gamma(t)$. Thus γ is smooth.

For the second part, let $\gamma \in L^x\mathbb{R}$ be a smooth loop. The associated map $S^1 \to L^x\mathbb{R}$ factors as $S^1 \to L\mathbb{R} \to L^x\mathbb{R}$. The first factor is smooth by assumption whilst the second is a continuous linear map and hence smooth. □

Thus although for continuity there is not much to choose between $L\mathbb{R}$ and $L^0\mathbb{R}$, once we get to smoothness we easily see the difference.

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