

Projective properties of certain orthogonal arrays

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SUMMARY

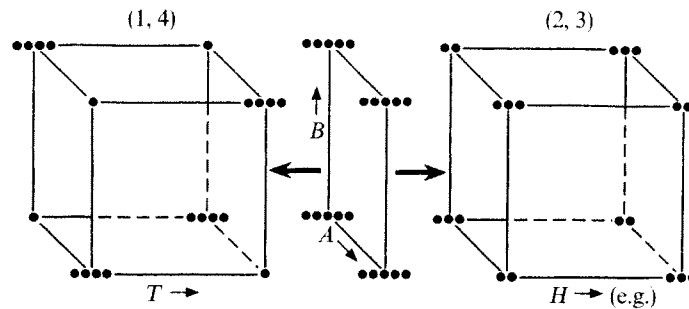
A question of importance in factor screening is when a two-level orthogonal design for a multifactor experiment can be projected into lower dimension, typically $P = 2$ or 3 . New results relate to the projectivity P of saturated designs in which $n - 1$ factors are tested in n runs. It is shown that: a design obtained by 'doubling' an $n \times n$ orthogonal array is always of projectivity $P = 2$; a two-level cyclic design is either a factorial array, and hence has $P = 2$, or it has $P = 3$; a two-level orthogonal design with $4m$ runs, m odd, has $P = 3$. In particular these results allow the designs derived by Plackett & Burman (1946) to be categorised in terms of these projective properties.

Some key words: Design projectivity; Design resolution; Factor screening; Fractional factorial design; Orthogonal array; Plackett–Burman design.

1. INTRODUCTION

At the preliminary stage of an experimental investigation a hypothesis of factor sparsity is often appropriate. That is to say, of a larger number k of factors to be tested only a small subset, typically 2 or 3, are expected to be active. An active factor is one that individually or interactively produces change in the response. A factor which is not active is said to be inert. This means that no main effect or interaction involving that factor occurs. Screening designs are then needed to identify the active subset.

An important source of screening designs are the two-level orthogonal arrays. See, for example, Plackett & Burman (1946), Rao (1947), M. J. Hall, in Jet Propulsion Laboratory Research Summary 36–10 'Hadamard matrices of order 16', and in Jet Propulsion Laboratory Technical Report 32–761 'Hadamard matrices of order 20', Raghavarao (1971), Hedayat & Wallis (1978). An orthogonal two-level array H_n is a $n \times n$ matrix with orthogonal columns, where $n = 4m$ with m a positive integer, and with a first column of $+1$'s. The remaining $n - 1$ contrast columns thus consist of an equal number of $+1$'s and -1 's. Arrays obtained by renumbering rows, renumbering contrast columns, or switching all the signs in a contrast column are regarded as equivalent. An orthogonal array can be used to generate a statistical design by associating k of its contrast columns with the levels of k experimental factors. In this paper we consider in particular designs for which $k = n - 1$. The design is then sometimes called a saturated design (Box & Wilson, 1951) or a main effect plan, as given by S. Addelman and O. Kempthorne in Arlington Hall Station ASTIA reports 'Orthogonal main-effect plans', pp. 220, 224, 226. When interactions between factors must be taken into account the extensive aliasing of effects that results has sometimes led to questions as to the practical usefulness of such designs. However it was argued (Box & Hunter, 1961; Box, Hunter & Hunter, 1978, p. 338) that the appropriate rationale for the use of designs for factor screening was their projection properties. Since every pair of contrast columns must contain adjacent elements of the form $(- -)$, $(+ -)$, $(- +)$ or $(+ +)$ each replicated m times, the design 'projects' m replicates



<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>J</i>	<i>K</i>	<i>L</i>	<i>M</i>	<i>N</i>	<i>O</i>	<i>P</i>	<i>Q</i>	<i>R</i>	<i>S</i>	<i>T</i>
+	+	+	+	-	-	-	+	+	-	-	+	-	-	+	+	-	+	-	-
+	+	+	-	+	-	-	-	+	+	-	-	+	-	+	-	+	-	+	-
+	+	+	-	-	+	-	+	-	-	+	-	+	+	-	+	-	-	+	-
+	+	+	-	-	-	+	-	-	+	+	+	-	+	-	-	+	+	-	-
+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	+
+	+	-	+	-	-	+	-	+	+	+	-	-	+	+	+	+	-	+	+
+	+	-	-	+	+	-	+	-	+	+	+	-	+	-	+	+	-	-	+
+	+	-	+	+	-	-	+	+	-	+	-	+	+	-	-	+	+	-	+
+	+	-	-	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	-	+	-	+	-	+	+	-	+	-	-	+	+	+	+	+	+	-	+
+	-	+	-	+	+	-	-	+	-	+	+	-	+	+	-	-	+	+	+
+	-	+	+	-	-	+	+	-	-	+	+	+	-	+	+	+	-	+	+
+	-	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	-	-	-	+	-	+	-	+	-	+	+	+	-	-	+	-	-	-	-
+	-	-	-	-	+	+	+	+	+	-	-	-	+	+	-	+	-	-	-
+	-	-	+	-	+	-	-	-	+	+	-	+	-	+	-	-	+	-	-
+	-	-	+	+	-	-	+	-	+	-	+	-	+	-	-	-	-	+	-
+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	+	+	+	+	+

Fig. 1. Rearrangement of an H_{20} orthogonal array given by Plackett & Burman (1946) partitioned into four subarrays by the signs of arbitrarily chosen columns labelled *A* and *B*. Addition to columns *A* and *B* of any further column such as *H* chosen from columns *C*–*S* results in a (2, 3) projection. Addition of column *T* produces a (1, 4) projection.

of a 2^2 factorial design in every pair of factors and hence we shall say it is of projectivity P at least 2. More generally we define the projectivity of a design as follows.

DEFINITION. A $n \times k$ design D with n observations and k factors each at 2 levels will be said to be of projectivity P if it is such that every subset of P factors out of the possible k contain a complete 2^P factorial design, possibly with some points replicated. The resulting design will then be called a (n, k, P) screen.

For preliminary illustration see Fig. 1. This shows an H_{20} orthogonal array which produces a $(20, 19, 3)$ screen.

2. TWO IMPORTANT CLASSES OF TWO-LEVEL ORTHOGONAL ARRAYS

When $n = 2^r$, one type of $n \times n$ orthogonal array H_n can be generated from the corresponding 2^r factorial design by writing down a column of +1's, denoted by I , followed by the r columns $c_1, c_2, \dots, c_i, \dots, c_j, \dots, c_r$ of ± 1 's of the complete 2^r factorial design. Now represent by $c_h = c_i c_j$

the operation whereby c_h is an entry-wise product of columns $c_i c_j$, and in particular $c_i c_i = I$. A further $n - r - 1$ columns corresponding to the interaction columns of the original factorial can be obtained from all possible products of the individual columns, thus

$$c_{r+1} = c_1 c_2, \quad c_{r+2} = c_1 c_3, \quad \dots, \quad c_{n-1} = c_1 c_2, \dots, c_r. \quad (1)$$

The resulting $n \times n$ matrix will be called a factorial orthogonal array and, following Finney (1945), the saturated design produced by its $n - 1$ contrast columns is a $2^{-(n-r-1)}$ fraction of a 2^{n-1} factorial with the $n - r - 1$ generating relations given by the identities (1). These may be conveniently rewritten with I on the left and the generating 'word' on the right as

$$I = c_1 c_2 c_{r+1}, \quad I = c_1 c_3 c_{r+2}, \quad \dots, \quad I = c_1 c_2, \dots, c_r c_{n-1}. \quad (2)$$

Multiplying these generators together in all possible ways produces the defining relation for the fractional factorial with I on the left and $2^{n-r-1} - 1$ words of the identity on the right, from which the alias relationships between the effects can be constructed. Any other fractional factorial containing $k < n - 1$ factors may be derived by omitting columns from the saturated factorial array and the defining relation for this derived design is obtained by omitting all words containing dropped factor symbols.

In a paper by Box & Hunter (1961) a criterion given for the classification of fractional factorials was their resolution R , defined as the length of the shortest word in the defining relation. These authors also stated, in effect, that a design of resolution R was of projectivity $P = R - 1$. To see this consider a parent fractional factorial design D_k of resolution R whose k columns are associated with k factors. The words in the defining relation of any fractional factorial D_P derived by dropping all but P columns from D_k are a subset of those of D_k . Therefore D_P is of resolution R or greater. But the longest word corresponding to a main effect or interaction of the factors in D_P is of length P . So if P is less than R , D_P has no defining relation and its effects have no aliases and is therefore a 2^P factorial possibly replicated. In particular, if $P = R - 1$ then any choice of P factors from the original k yield a 2^P factorial design, possibly replicated. Thus from (2) any saturated design derived from a factorial array has $R = 3$ and $P = 2$. Designs of projectivity greater than 2 can be obtained from factorial arrays but only for screening fewer factors. For example a $(n, \frac{1}{2}n, 3)$ screen can always be obtained by dropping from the factorial array the $\frac{1}{2}n - 1$ contrast columns containing an even number of letters in the generating factorial design. But this increase in projectivity from 2 to 3 is only obtained at the cost of reducing the number of screened factors from $n - 1$ to $\frac{1}{2}n$. Thus a 16 run factorial orthogonal array can be used to screen 15 variables at projectivity 2 but only 8 variables at projectivity 3.

Whilst factorial arrays exist only when n is a power of 2, other two-level orthogonal arrays are available when $n = 4m$ and m is an integer. In particular, Plackett & Burman (1946) derived orthogonal arrays for all such cases when $n \leq 100$, with one single exception. By analogy with the saturated fractional factorials it had been conjectured that these designs would provide $(n, n - 1, P)$ screens with projectivity only 2. However a computer search by Bisgaard in 1987 (Box & Bisgaard, 1993) showed that such designs could be of projectivity higher than 2 and in particular, that every one of the 165 three-dimensional projections of the $k = 11$, $n = 12$ was a full 2^3 factorial design with four replicated runs which themselves formed a half-replicate, main-effect plan. Thus for the purpose of screening the 12 run design appears to do better than the 16 run factorial array. Further computer enumeration by Lin & Draper (1992) showed that similar results were possible for some, but not all, of the remaining Plackett & Burman designs. They referred to a projection in which half the vertices of the projected factorial cube were replicated t times and the remainder of the vertices $s = m - t$ times as a (t, s) projection. Thus each of the 165 three-dimensional projections of the 12-run orthogonal array design was a $(1, 2)$ projection. Also Fig. 1 illustrates, for the 20 run orthogonal array given by Plackett & Burman, how both $(2, 3)$ and $(1, 4)$ projections occur. Computer studies of other aspects of the projective rationale for orthogonal arrays have been made by J. C. Wang and C. F. J. Wu, in the University of Waterloo IIQP report RR93.08 'A hidden projection property of Plackett-Burman and related designs', and by Lin (1993).

3. SOME GENERAL RESULTS

This paper gives three general results which provide a theoretical basis for these empirical discoveries and in particular can be used to categorise the projective properties of designs given by Plackett & Burman (1946). We first note that one way in which these authors obtained an orthogonal array for $2n$ runs was by 'doubling' an orthogonal array for n runs as follows:

$$H_{2n} = \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}.$$

PROPOSITION 1. *A saturated design obtained from a doubled $n \times n$ orthogonal array is always of projectivity $P = 2$ and only 2.*

Proof. If the contrast columns in H_{2n} are denoted by $c_1, \dots, c_{2n-1}$ then, noting that the first n elements of c_n are +1's and the remainder are -1's, it follows for instance that $c_i c_n = c_{n+i}$, and hence that $I = c_i c_n c_{n+i}$. Thus a design formed from these columns can only have rows with sign combinations $(--+)$, $(+-)$, $(-+-)$, $(+++)$, and consists of $n/2$ replicates of a half fraction of the 2^3 factorial. The design is therefore of projectivity $P = 2$ and only 2. $\square$

As a special case of these results it follows once more that every saturated fractional factorial is of projectivity $P = 2$, since, as pointed out by Plackett & Burman (1946) these designs may also be obtained by doubling. However it is not true that, with $n = 4m$ and m even, the corresponding saturated design is always of projectivity only 2. For example, while the H_{24} obtained by doubling yields a saturated design with $P = 2$, that obtained by Plackett & Burman using cyclic generation has $P = 3$.

Another way of obtaining orthogonal arrays is by cyclic generation. For a cyclic orthogonal array H_n it is possible to write down all the $n - 1$ contrast columns knowing only the sequence of signs to be applied in the first row. The cyclic 12-run orthogonal array may, for instance, be written down knowing only the 11 signs in the first row sequence $(++-+++-+--)$. Shifting this row cyclically one place 10 times and adding a final row of minus signs and an initial column of plus signs produces a 12×12 orthogonal array.

PROPOSITION 2. *A saturated design obtained from a cyclic orthogonal array is either a factorial orthogonal array with $P = 2$ and only 2, or else has projectivity $P = 3$.*

Proof. Recalling that any two contrast columns are regarded as identical if one can be obtained from the other by sign reversal, it follows from the cyclic generation property that, if $c_i c_{i+j} = c_l$ is in the design, the entry-wise product of any two columns at distance j must be in the design. Hence $c_i c_{i+2j} = c_i c_{i+j} c_{i+j} c_{i+2j} = c_l c_{l+j}$ is in the design and more generally $c_i c_{i+hj}$ for $h = 1, 2, \dots$ is in the design. In particular, if $c_1 c_2$ is in the design, the entry-wise product of any two columns is in the design.

Now suppose $c_1 c_2$ is not in the design. Then since c_1 will be the column that follows after c_{n-1} in the cyclic generation procedure, it follows that $c_{n-1} c_1$ is not in the design either. But the distance between c_1 and c_{n-1} is an even number, $n - 2$, from which it follows that no entry-wise product of columns with distance 2, or in general with distance $2h$ ($h = 1, 2, \dots$), can be in the design. If $c_1 c_{1+j}$ is a column in the design for j odd then $c_{n-1} c_j$ must also be in the design. But since $n - (j + 1)$ is an even integer, that is not possible. Hence, if $c_1 c_2$ is not in the design no entry-wise product of any two columns is in the design.

Now assume a design constructed by cyclic generation contains $c_1 c_2$ and thereby all of their two-factor interaction columns. If such a design contains more columns than c_1 , c_2 and $c_1 c_2$, it must contain a column c_3 orthogonal to these, and since it also contains all the possible two-factor interaction columns it must also contain $c_1 c_3$, $c_2 c_3$ and $c_1 c_2 c_3$, and none of these columns can be equal to any of the first three or to a column containing only +1's. By induction the same argument now gives us that if a design generated by cyclic generation contains $c_1 c_2$ it must have $2^r - 1$ contrast columns for some r and therefore must be a factorial orthogonal array with projectivity $P = 2$ and only 2. $\square$

PROPOSITION 3. Any saturated two-level design obtained from an orthogonal array containing $n = 4m$ runs, with m odd, is of projectivity $P = 3$.

Proof. Consider a particular orthogonal array for which m is odd and let u be a column vector of m ones. Then I is a column of $n = 4m$ ones which can be written in the partitioned form $(u, u, u, u)'$. Also, the rows of the orthogonal array can be re-ordered so that the array is partitioned into four $m \times n$ subarrays in which two arbitrarily chosen contrast columns c_i, c_j are partitioned in the form $(u, u, -u, -u)'$ and $(u, -u, u, -u)'$.

Now consider any other arbitrarily chosen column $c_h = (c_{h1}, c_{h2}, c_{h3}, c_{h4})'$ after this re-ordering of rows. Orthogonality implies that c_h satisfies the linear equations $c_h' I = c_h' c_i = c_h' c_j = 0$, hence

$$u'c_{h1} = -a_h, \quad u'c_{h2} = +a_h, \quad u'c_{h3} = +a_h, \quad u'c_{h4} = -a_h.$$

Now let t_h represent the number of plus signs in c_{h1} and in c_{h4} , or equivalently the number of minus signs in c_{h2} and in c_{h3} . Then if $s_h = m - t_h$, we have $a_h = s_h - t_h$ and, if necessary by switching signs in the whole column c_h , a_h can be taken to be positive, so that $t_h < s_h$. Also since m is odd, a_h cannot be zero; and it cannot be m , for then c_h would correspond to the contrast c_i, c_j , and no additional column could be simultaneously orthogonal to c_i, c_j and c_h . Thus, in the arbitrarily chosen space of c_i, c_j and c_h the projected experimental points will lie on the vertices of a cube, and will be distributed as shown in Table 1.

Table 1. *Distribution of projected experimental points*

Replicated t_h times			Replicated s_h times		
—	—	+	—	—	—
+	—	—	+	—	+
—	+	—	—	+	+
+	+	+	+	+	—

The projected design will therefore consist of a full 2^3 factorial replicated t_h times with an additional half-replicate having defining relation $I = c_i c_j c_h$ replicated $a_h = s_h - t_h$ times. $\square$

Note that different choices of the columns c_j, c_k, c_h can result in different combinations of t and s and hence in different amounts of replication of the two design parts. This proposition is in agreement with a recently published result of Cheng (1995) who bases his proof on general properties of orthogonal arrays of a given strength.

Figure 1 shows a rearrangement of the 20×20 orthogonal array given by Plackett & Burman (1946), used earlier for preliminary illustration, with columns associated with factors $A, B, C, \dots, T$ partitioned into four subarrays obtained by setting $A = c_i, B = c_j$ respectively. Since the original design is obtainable by cyclic generation and also since $m = 5$ is odd, the design is a $(20, 19, 3)$ screen with projections either of type $(2, 3)$ or of type $(1, 4)$. The latter are generated by a single column with four like signs and one unlike sign in each quadrant: for the case illustrated this is column T . Thus of the 969 three-dimensional projections, only $\frac{1}{17}$ are of the $(1, 4)$ type and $\frac{16}{17}$ are of the $(2, 3)$ type. Notice, however, that the diagram does not imply that we can omit column T to produce a $(20, 18, 3)$ screen for which all the projections are of the $(2, 3)$ type. This is because a different choice of two columns c_j, c_k to define the four subarrays would produce a different $(1, 4)$ column. When r_h and s_h are each at least equal to two an additional analysis for dispersion effects is facilitated: see, for example, Box & Meyer (1986a).

The arrays tabled by Plackett & Burman are not, of course, exhaustive. Other orthogonal arrays of potential interest to the practitioner exist. For example, 5 distinct orthogonal arrays are given by M. J. Hall in the unpublished summary mentioned in § 1 for the case $n = 16$. All of these cases are $(16, 15, 2)$ screens. But we have found it possible to show that one of them produces a $(16, 14, 3)$ screen when a particular column is dropped and also that the three other arrays that are different

Table 2. Projectivity P of Plackett & Burman (1946) designs for $n \leq 84$

		n																			
		8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	78	80	84
		m																			
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$P = 3$:																					
m odd			*		*		*		*		*		*		*		*		*		*
cyclic			*		*	*			*		*	*			*		*	*		*	*
$P = 2$		*		*				*		*				*		*					

$P = 3$ cyclic designs are not factorial. $P = 2$ designs are factorial arrays or obtained by doubling.

from the 16-run factorial array can produce (16, 12, 3) screens when particular sets of three columns are dropped. We plan to discuss these and other designs in a separate paper.

To illustrate the usefulness of the three main results of this paper we show in Table 2 how they characterise the projectivity of each of the Plackett & Burman designs for $n \leq 84$. Of the 20 saturated designs considered, 14 are of projectivity 3, four are factorial arrays of projectivity $P = 2$ and two obtained by doubling are also of projectivity $P = 2$.

Some discussion of the analysis of orthogonal array designs under the hypothesis of effect sparsity is in Box & Meyer (1986b, 1993).

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