Representation theorems for transfinite computability and definability

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1 Introduction

The continuous functionals were independently introduced by Kleene [6] and Kreisel [7]. Kleenes motivation was to isolate a subclass of the total functionals of pure, finite type closed under S1-S9-computations (Kleene [5]) which could be described using countable information. He defined his functionals via the *associates* and the idea is that an associate for a functional Ψ contains sufficient information about Ψ to decide $\Psi(F)$ uniformly in any associate for F. Kleene called the functionals countable.

Kreisel's continuous functionals are defined via sets of formal neighbourhoods, and the definition is much in the spirit of what is now known as domain theory. Kreisel actually defines the hierarchy of partial continuous functionals, and then use the hereditarily total ones in his hierarchy. Kreisel's motivation was to analyse the concept of *constructivly true*. It was clear already at the time of publication that the two definitions were basicly equivalent.

Following Kleene's motivation it is established that the continuous functionals support a rich computation theory, see e.g. Normann [12, 15]. In this paper we will pursue some of the ideas in Kreisel's approach and show that transfinite versions of the continuous functionals can be used to represent complex properties. We will be more precise later.

One basic result proved both by Kleene and Kreisel is the density theorem. They formulated it in different ways. Kleene showed that the set of finite number-sequences approximating an associate is primitive recursive, while Kreisel showed that the total functionals is dense in the set of partial functionals. It is the extension of the density theorem to transfinite types and the fine-structure constructed as a part of the proof that will be our main tool in this paper. This is again based on work by Berger [1, 2, 3, 4], Kristiansen and Normann [9] and Normann [14].

The Kleene-Kreisel continuous functionals has been extended to various transfinite and non-wellfounded types. In Kristiansen and Normann [9] a wellfounded hierarchy of domains with totality was constructed. The base types of this hierarchy are the set of natural numbers and the set of Boolean values, and the hierarchy is closed under the formation of certain dependent sums and products of continuously parameterised families of domains with totality. A central part of the construction in [9] is the use of a domain S of syntactic forms where the wellfounded types are represented by a subset S_{wf} of S. In addition there is an interpretation $I(s)$ of s as a domain with totality for each $s \in S_{\mathrm{wf}}$. For this hierarchy the density theorem holds: If $s \in S_{\mathrm{wf}}$ then the total elements are dense in $I(s)$

In order to prove the density theorem we have to develope a concept dual to density. In [9] we used the so called h-functions. An h-function on a domain X with totality is a continuous map $h : X \to (\mathbb{N} \to \mathbb{N})$ such that $h(x)$ is total whenever x is total, and $h(x) = h(y)$ if and only if $x = y$ for total x and y. In the proof we prove effective density and the existence of h-functions simultanously.

In [10] the construction from [9] is extended to cover types defined by strictly positive or generally positive induction. In the case of strictly positive induction we can represent the types defined by elements in the domain S, we simply extend the set of s for which we can define the total elements of $I(s)$. In the case of generally positive induction we need a variant where we somehow can distinguish between positive and negative occurences of sums and products. In general, density will not hold for inductively defined types, we may define the empty type by induction. However, sufficient criteria are given in [10] for when density holds.

One drawback of the constructions in [9, 10] is that they are very concrete and there is little trace of the conceptual analysis of totality initiated by Normann in [13].

Berger [1, 2] initiated independently an alternative conceptual analysis of totality. He defined the dual of density as a property of totality, the total objects should all respond to some dense set of Boolean valued tests. In

Berger [3] this analysis is carried further to parameterisations of domains with totality, and in [4] the analysis is extended to universe operators. In [3] he proves a general density theorem that will cover the results from [9, 10]. As we will see, we may recover a system of h-functions from Berger's totality-property.

The conceptual analysis is also central in Kristiansen [8] where similar results are developed for qualitative domains and coherence spaces.

The main concept of this paper is that of representation via totality in a domain. The application in this paper will be that of comparing complexity. In Normann [14] we characterised the complexity of the hierarchy of well-founded types with totality as the one of semicomputability in ${}^3E$. In this paper we show that the complexity of the hierarchy of domains with totality defined by iterated positive induction in Kristiansen and Normann [10] is the same as that of computations relative to a certain normal functional J of type 3. This functional corresponds to iterated positive induction over Baire-space.

A further application of the representation technique is that given a representation of a relation we may develope the logic of that relation as a proof-system using transfinite proofs with continuous branchings. These logic-systems are defined and explored in Normann [16].

In section 2 we will define the basic notions and isolate the basic methods. Many of them are essentially due to Kreisel [7].

In section 3 we will see how these methods can be used to represent subsets of $\mathbb{N}^{\mathbb{N}}$ defined by positive induction via totality in inductive domains. The construction and proof is a simple combination of the density-result for inductive domains and Kreisel's proof of the fact that the constructive and classical interpretations of statements of analysis not containing disjunction or existential quantifiers will be the same.

In section 4 we will extend the representation theorem to computations relative to the functional J.

In section 5 we will show how to represent structures of the form $L_\kappa(HC)$. In Normann [16], we see the developement of representation techniques as a part of a general program where we ad some fine structure to sets transfinitly definable over the continuuum. The methods of section 5 are general, whenever we obtain representations for some notion of transfinite computability, these methods provide us with representations for the corresponding initial segments of $L(HC)$. Thus these methods relate representation theorems for transfinite computations to the aims of the general program.

2 Parameterisation and representation

2.1 Parameterisations

Kreisel [7] used the continuous functionals to give an interpretation of the constructive meaning of statements of analysis or second order number theory. Each statement $\Phi(\vec{x})$ is transcribed to a statement $\exists y^\sigma \forall z^\tau \Phi^c(\vec{x}, y, z)$, where σ and τ are finite types. If we interpret σ and τ classically, we get the classical interpretation of $A(\vec{x})$, while if we let y^σ range over the *recursive* functionals of type σ, and z^τ range over the *continuous* functionals of type τ, we get Kreisel's constructive interpretation. Here *recursive* means that the object has a computable associate, while *computable* will mean that it is obtainable via Kleene's S1 - S9 (Kleene [5]). Kreisel shows that if $\Phi(\vec{x})$ does not contain disjunctions or existential quantifiers, then the constructive and classical interpretations of $\Phi(\vec{x})$ are equivalent. This is of course proved classically. The essence of Kreisel's proof is that for statements of the above form, we uniformly compute a partial object $\beta(\vec{x})$ of type σ such that $\beta(\vec{x})$ is total if and only if $\Phi(\vec{x})$ is true, and in this case we have $\forall z^\tau \Phi^c(\vec{x}, \beta(\vec{x}), z^\tau)$. In [14] we formulated the representation theorem for ${}^3E$ exactly in this way.

We will assume familiarity with domain theory as presented in e.g. Stoltenberg-Hansen & al. [17]. We will use $\mathbb{N}$ to denote the set of natural numbers, $\mathbb{B}$ to denote the set of Boolean values *true* and *false*, and $\mathbb{O}$ denote the singleton set $\{0\}$. We let $\mathbb{N}_\perp$, $\mathbb{B}_\perp$ and $\mathbb{O}_\perp$ denote the corresponding flat domains.

Definition 1 If X is a domain and $\bar{X} \subseteq X$, we say that

a) $\bar{X}$ is *dense* if every compact $x_0 \in X$ can be extended to an element of $\bar{X}$.

b) $\bar{X}$ is *co-dense* if whenever A is an unbounded finite set of compacts in X there is a continuous map $t : X \to \mathbb{B}_\perp$ such that $t : \bar{X} \to \mathbb{B}$ and $t : A \to \mathbb{B}$, but t is not constant on A.

Remark Berger [1, 2] isolated the co-density property and viewed it as a property of totality. He showed that the class of domains with totality satisfying density and co-density are closed under the function-space construction.

We will work with parameterised families of domains with totality satisfying density and co-density in a uniform way. General notions of uniform density and co-density are defined in Berger [4].

We will use the following notation:
A parameterisation will be denoted (X, F) where X is a domain and F is a functor from X seen as a category into the category of domains commuting with direct limits.
When we write $(\bar{X}, \bar{F}) \sqsubseteq (X, F)$ we mean that $\bar{X} \subseteq X$ and $\bar{F}(x) \subseteq F(x)$ for each $x \in \bar{X}$. The category of domains is defined in e.g. [17].

A domain is *separable* if the set of compacts is countable. We have

Lemma 1 **a)** *Let X be separable and let $\bar{X} \subseteq X$ be dense and co-dense. There is a continuous function $h : X \to (\mathbb{N}_\perp \to \mathbb{B}_\perp)$ such that*

- **i)** *If $x \in \bar{X}$ then $h(x)$ is total*
- **ii)** *If x and y are in $\bar{X}$, then $h(x) = h(y)$ if and only if $\{x, y\}$ is bounded in X.*

b) *If (X, F) is a parameterisation of separable domains over a separable domain with a total sub-parameterisation $(\bar{X}, \bar{F}) \sqsubseteq (X, F)$ satisfying uniform density and co-density, then we may construct $h_x : F(x) \to (\mathbb{N}_\perp \to \mathbb{B}_\perp)$ as above uniformly in $x \in \bar{X}$.*

Proof
We prove a) and notice that Berger's notion of uniformity of parameterisations is sufficient to ensure that the construction of h is uniform. Let $\{(x_i, y_i)\}_{i \in \mathbb{N}}$ be an enumeration of all unbounded pairs of compacts. Let t_i be a total Boolean-valued function separating x_i and y_i. Let $h(x)(i) = t_i(x)$. It is easy to see that this construction works.

Remark These h-functions give us an oportunity to compare total elements. In general $h^{-1}(\mathbb{N} \to \mathbb{B})$ will be much larger than $\bar{X}$, but this inverse

will carry many of the fine-structure properties of the total elements. This set of course depends on the selection of the sequence $\{t_i\}_{i\in\mathbb{N}}$. In the proof of the combined density and co-density theorem in Berger [3, 4] there is a precise construction of the separating boolean-valued functions as well as of the total extensions of the compacts, and these constructions are deeply interlocked.
The h-functions of this and previous papers, as well as the separating tests of Berger [1, 2, 3], may seem a bit ad hoc, but we claim that they are not. Kreisel [7] used the enumeration of a dense set of total objects in proving the equivalence between the classical and constructive interpretation of certain statements. The h-function can be seen as the result of evaluating an object along an enumerated dense set of possible evaluation paths down to atomic values.

Before proceeding we need to reveal a few details about the main result of Berger [3] and about the construction leading to it.

A parameterisation (S, F) with totality $(\bar{S}, \bar{F})$ is called a $\Pi\Sigma$-system if:

i) Continuously in $s \in \bar{S}$ we can decide if $F(s)$ is a base type (which will be a flat domain), a Π-type or a Σ-type.

ii) If $F(s)$ is a Π-type or a Σ-type, we can continuously in $s \in \bar{S}$ find $i(s) \in \bar{S}$ and $\pi(s) : \bar{F}(i(s)) \to \bar{S}$ such that

$$F(s) \approx \Pi(x \in F(i(s))F(\pi(s)(x))$$

or

$$F(s) \approx \Sigma(x \in F(i(s))F(\pi(s)(x))$$

(where $\approx$ means 'isomorphic').

A $\Pi\Sigma$-system is *ranked* if we can associate a number to each compact, and this ranking is monotone with respect to the construction of new compacts in Π- and Σ-domains.

In Berger [4] $\Pi\Sigma$-systems are not defined explicitly, but will be special cases of fixpoints of operators on domains and parameterisations.

Proposition 1 (Berger [3])
If (S, F) is a ranked $\Pi\Sigma$-system and if $\prod(s \in \bar{S})\bar{F}(s)$ contains one element, then $(F(s), \bar{F}(s))$ satisfies density and co-density uniformly in s for $s \in \bar{S}$.

As a consequence of the proof each domain is equipped with a "density", i.e. selected objects extending each compact, and with a "co-density", i.e. selected Boolean valued tests separating separable sets of compacts. We call the elements of these sets for *density witnesses* and *co-density witnesses* respectivly. An object x is called *weakly total* if $t(x) \in \mathbb{B}$ for all co-density witnesses t. Then $h(x)$ is total exactly when x is weakly total.

The following properties of density witnesses and co-density witnesses can be extracted from the density-proof in Berger [3, 4]:

Proposition 2

1. If d is a density witness in $\Pi(x \in X)Y_x$ and $x \in X$ is weakly total, then $d(x)$ is a density witness in Y_x.

2. If t is a co-density witness in $\Pi(x \in X)Y_x$ then there is a density witness d in X and a co-density witness t' in Y_d such that for all f, $t(f) = t'(f(d))$. We will without causing any harm assume that all t constructed from a density and co-density witness in this way will be a co-density witness.

3. A density witness in a function space $X \to \mathbb{N}$ will be bounded.

4. A co-density witness for $\mathbb{N}_\perp$ will be of the form

$$T_n(m) = true \text{ if } n = m$$
$$T_n(m) = false \text{ if } n \neq m$$

Similar properties are satisfied by the more ad hoc constructions in [9].

From now on we will let $(\bar{S}, \bar{F}) \sqsubseteq (S, F)$ be a fixed, ranked $\Pi\Sigma$-structure with one uniform, total object in $\Pi(s \in S)F(s)$. We will let C be the subcategory of the category of domains consisting of all $F(s)$ for $s \in S$. We call $X \in C$ a *Domain with totality* if $X = F(s)$ for some $s \in \bar{S}$, and we then write $\bar{X}$ for $\bar{F}(s)$. We let $\bar{C}$ denote the set of domains with totality.

A parameterisation $\{Y_x\}_{x \in X}$ from C will always be of the form $Y_x = F(G(x))$ where $G : X \to S$ is continuous. We will supress G in this notation. The parameterisation is *total* if $X \in \bar{C}$ and $Y_x \in \bar{C}$ whenever $x \in \bar{X}$.

We will assume that $\bar{C}$ contains copies of $\mathbb{O}_\perp$, $\mathbb{B}_\perp$ and $\mathbb{N}_\perp$ and that C and $\bar{C}$ are closed under dependent sums and products in a uniform way.

In the proof from Berger [3], the density and co-density witnesses will be defined for all $X \in C$. We let C_E be the largest subset of C that is a $\Pi\Sigma$-structure, and where all co-density witnesses will be total on all density witnesses. We will only work with parameterisations $\{Y_x\}_{x\in X}$ where $X \in C_E$ and $Y_x \in C_E$ whenever x is weakly total in X (The definition of 'weakly total' naturally extends to C_E). It is easy to see that C_E is closed under dependent sums and products of C_E-parameterisations.

2.2 Representations

From now on we will let X, Y etc. denote domains in C_E, and all parameterisations will be C_E-parameterisations. Thus all our concepts will be relative to the category C and the subsets C_E and $\bar{C}$. The system of density and co-density witnesses must be seen as a part of this structure.

Definition 2 Let $X \in \bar{C}$, $A \subseteq \bar{X}$.

a) A *weak representation* of A is a pair $\{Y_x\}_{x\in X}$ and $\beta \in \Pi(x \in X)Y_x$ such that for $x \in \bar{X}$

$$x \in A \text{ if and only if } Y_x \in \bar{C} \text{ and } \beta(x) \in \bar{Y}_x.$$

b) A weak representation is a *representation* if

1. $\beta(x)$ is weakly total for all weakly total x.
2. If $\beta(x)$ is not total, and $y \in Y_x$ is total there is a co-density witness that separates $\beta(x)$ and y.
3. $\beta(x)$ is not a density witnesss.

c) A representation $\{Y_x\}_{x\in X}$, β of A is *full* if $Y_x \in \bar{C}$ for all $x \in \bar{X}$.

In Normann [14] the main theorem is a representation theorem in this sense, we reduce the set of terminating computations relative to ${}^3E$ to the parameterisation $(S_{\text{wf}}, \lambda s \in S_{\text{wf}}.I(s)_{\text{tot}})$.

We will now show how to transform the representation of one set to a representation of another set defined from the first one. The two first constructions are based on the methods of Kreisel [7]

Construction 1 Let $X \in \bar{C}$ and let β be a full representation of a set $A \subseteq \bar{X}$ into $\{Y_x\}_{x \in X}$. Define $\{Z_x\}_{x \in X}$ and γ by

$$Z_x = Y_x \to \mathbb{N}_\perp$$

and

$$\gamma(x) = \lambda y \in Y_x.\mu k(h_{Y_x}(y)(k) \neq h_{Y_x}(\beta(x))(k)).$$

Lemma 2 *The result of construction 1 will be a representation.*

Proof
First let us see that γ is a weak representation for the complement of A. Let $x \in \bar{X}$.
Assume that $x \in A$. Then $\beta(x) \in \bar{Y}_x$ and $\gamma(x)(\beta(x))$ is not terminating, so $\gamma(x)$ is not total.
Now assume that $x \notin A$. Then $y \neq \beta(x)$ for all y in $\bar{Y}_x$. By assumption, $h_{Y_x}(\beta(x))$ is total and different from $h_{Y_x}(y)$ for all total $y \in Y_x$, and γ is total.
We now prove the three extra properties of a representation. The property numbers refer to Proposition 2.

1. Since $\beta(x)$ is not a density witness, $\gamma(x)$ will be total when restricted to density witnesses. Then all co-density witnesses will terminate for $\gamma(x)$.

2. Let $x \in A$, i.e. $\gamma(x)$ is not total. Let $z \in Z_x$ be total. Let $\{y_i\}_{i \in \mathbb{N}}$ be a sequence of density witnesses in Y_x converging to $\beta(x)$. Now $z(\beta(x)) = \lim_{i \to \infty} z(y_i)$ while $\lim_{i \to \infty} \gamma(x)(y_i) = \infty$. It follows that $\gamma(x)(y_i) \neq z(y_i)$ for some i. By properties 2. and 4. there is a co-density witness t defined by $t(u) = T_n(u(y_i))$. This will separate $\gamma(x)$ and z.

3. $\gamma(x)$ is unbounded, so by property 3. $\gamma(x)$ is not a density witness.

Construction 2 Let $\{Y_{(x_1,x_2)}\}_{(x_1,x_2) \in X_1 \times X_2}$ and β be a representation of $A \subseteq \bar{X}_1 \times \bar{X}_2$.
Let $x_1 \in \bar{X}_1$ be fixed.
Consider $Z_{x_1} = \Pi(x_2 \in X_2)Y_{(x_1,x_2)}$. Let $\gamma(x_1) = \lambda x_2.\beta(x_1, x_2)$.

Lemma 3 *The result of construction 2 will be a representation.*

Proof
It is easy to see that γ will be a weak representation, so we focus on the extra properties.

1. Let d be a co-density witness. Then by property 2. $d(\gamma(x_1)) = d'(\beta(x_1, e))$ for some totality witness d' and density witness e. $\beta(x_1, e)$ is weakly total so $d(\gamma(x_1))$ terminates.

2. Assume that $\gamma(x_1)$ is not total, but y is total. Then there is an x_2 for which $\beta(x_1, x_2)$ is not total. There is a co-density witness d' that separates $y(x_2)$ and $\beta(x_1, x_2)$.
Then there is a compact $u \sqsubseteq x_2$ such that d' separates $y(u)$ and $\beta(x_1, u)$. Let e be a density witness extending u. Then by property 2. d defined by $d(z) = d'(z(e))$ is a co-density witness that separates y and $\gamma(x_1)$.

3. We use property 1. $\gamma(x_1)$ is not a density witness because $\gamma(x_1)(x_2) = \beta(x_1, x_2)$ is not a density witness for any $x_2 \in \bar{X}_2$.

Construction 3 It is tempting to represent an atomic, true statement as a total element in $\mathbb{B}^{\perp}$ or $\mathbb{N}^{\perp}$. The drawback is that all total elements in these domains are density witnesses, and we will avoid that. Thus we will use the following representation of basic truth:

$$\text{Domain: } \mathbb{N}_{\perp} \to \mathbb{N}_{\perp}. \text{ Object: } \lambda k.k$$

Our representation of basic falsity will then simply be the representation of the negation of basic truth:

$$\text{Domain: } (\mathbb{N}_{\perp} \to \mathbb{N}_{\perp}) \to \mathbb{N}_{\perp}. \text{ Object: } \lambda f.\mu k.f(k) \neq k.$$

Construction 4 Our final example of how to construct new representations deals with certain bounded universal quantifiers. This construction goes beyond the methods of Kreisel [7]. The method first appeared in Normann [14] where it was used to handle composition in representing transfinite computations, see also Section 4 of this paper.

Let $\{Y_x\}_{x \in X}$ be a parameterisation of domains with totality. Let $B \subseteq \bar{X}$ be defined via the representation ϕ, i.e. $x \in B \Leftrightarrow \phi(x)$ is total in Y_x.
Let $\{Z_x\}_{x \in X}$ be a parameterisation from C_E and assume that $Z_x \in \bar{C}$ whenever $x \in B$.

Finally let $\psi \in \Pi(x \in X)Z_x$ be a representation of $R \subseteq \bar{X}$.

We will represent the statement $(\forall x \in B)R(x)$ as follows:
The domain will be $U = \Pi(x \in X)\Pi(y \in Y_x)Z_x(\mathrm{mod}\ y)$ where
$Z_x(\mathrm{mod}\ y) = Z_x$ if $h_{Y_x}(\phi(x)) = h_{Y_x}(y)$
$Z_x(\mathrm{mod}\ y)$ is the n'th approximation to Z_x as a domain with totality if $h_{Y_x}(\phi(x))(n) \neq h_{Y_x}(y)(n)$ for some minimal n.
The representing function will be

$$\xi = \lambda x \in A.\lambda y \in Y_x.\psi(x)(\mathrm{mod}\ y)$$

where we modify $\psi(x)$ to a total n'th approximation in $Z_x(\mathrm{mod}\ y)$ when the h-functions are different as above.

For this method to work we need that the n'th approximations are well defined. This can either be achieved by an explicit construction or via density of the domains with totality. We will return to this in the two applications of this construction.

Lemma 4 **a)** *The construction above will always give a domain with totality.*

b) *ξ is total if and only if the statement $(\forall x \in B)R(x)$ is true.*

c) *If $B \neq \emptyset$ or the representation of R is full, the construction gives a representation.*

Proof

a) Let $x \in \bar{X}$, $y \in \bar{Y}_x$.
If $h_{Y_x}(y) = h_{Y_x}(\phi(x))$ it means that $\phi(x)$ is total, so $x \in B$. Then $Z_x(\mathrm{mod}\ y) = Z_x$ is a domain with totality.
If $h_{Y_x}(y) \neq h_{Y_x}(\phi(x))$, then $Z_x(\mathrm{mod}\ y)$ is by construction a domain with totality.

b) Assume that ξ is total. Let $x \in B$. Then $\xi(x)(\phi(x)) = \psi(x)$ must be total, so $R(x)$ by the choice of ψ.
Conversly, if $R(x)$ then $\psi(x)(\mathrm{mod}\ y)$ will be total for all total y since $\psi(x)$ is total, so $\xi(x)$ is total for all total $x \in B$.
If $x \notin B$ then $\phi(x)$ is not total in Y_x, so $\xi(x)(\mathrm{mod}\ y)$ will be total in Z_x for all total $y \in Y_x$.

c) There are three properties to verify

1. ξ is weakly total: It is sufficient to show that for any density witness $x \in X$ and density witness $y \in Y_x$, $\xi(x)(y)$ is total in Z_x. This follows from the definition and the fact that $\phi(x)$ is not a density witness.

2. Let ξ not be total, $z \in U$ be total.
Since ξ is not total, there is an $x \in B$ such that $\psi(x)$ is not total. Then $\xi(x)(\phi(x)) = \psi(x)$ and $z(x)(\phi(x))$ is total, so $\psi(x)$ and $z(x)(\phi(x))$ can be separated by a co-density witness in Z_x. Using density witnesses close to $\phi(x)$ and x we can construct a co-density witness for U separating ξ and z.

3. ξ is not a density witness: Assume that ξ is total.
If $B \neq \emptyset$ we pick $x \in B$. Then $\xi(x)(\phi(x)) = \psi(x)$ which is not a density witness.
If $B = \emptyset$ we assumed that the representation of R is full over $\bar{X}$. Then pick any $x \in \bar{X}$. By the general properties, $\phi(x)$ is weakly total, so if ξ is a density witness, then $\xi(\phi(x)) = \psi(x)$ is also a density witness. This contradicts that ψ is a representation.

3 Representing positive induction

In this section we will consider a positive, inductive operator on $\mathbb{N}^{\mathbb{N}}$. We will use the following language:
Variables: x, y, z, etc. for functions in Baire space.
Terms: $t(x_1, \dots, x_n)$ where t is primitive recursive.
Predicates: $P(x_1, \dots, x_n)$ where P is primitive recursive.
Set variable: X
Connectives: $\neg$, $\wedge$ and $\forall$

We let $\Gamma(X, x)$ be a formula in the language described above, where the set variable X occurs positivly and x is the only free function variable. We aim to construct a family of domains with totality, indexed over $\mathbb{N} \to \mathbb{N}$ and a full representation of the least fixpoint of the operator induced by Γ.

In section 2 we saw how to transform a full representation for a set A to a full representation for the set $\{f \mid \Gamma(A, f)\}$. Our naive idea is to say that in order to find a full representation of an inductively defined set, we simply produce the least fix-point of the corresponding operator on the representation. We will then be faced with the following.

1. We have to construct the proper family of domains indexed by $\mathbb{N}_\perp \to \mathbb{N}_\perp$.

2. We have to identify the total objects in each domain.

3. We must prove totality and density for each domain involved.

4. We must prove that the final representation works.

1. will be easy, domain equations have solutions.
The total objects will be defined inductively, and this takes care of 2. inductively defined domains with totality are studied in Kristiansen and Normann [10], and our constructions may be covered by the constructions in [10]. Here we will use Berger's density theorem for ranked $\Pi\Sigma$-systems, see section 2. In order to satisfy Berger's requirements we will make some cosmetic changes in our representations. In all our constructions we will replace a product $\Pi(x \in X)Y_x$ by $\mathbb{O}_\perp \oplus \Pi(x \in X)Y_x$, and we will alter the definition of the representation accordingly. Then Berger's criterium will be trivially satisfied. This will take care of 3. We discuss and prove 4. after we have given the technical definition.

From now on we will let $\Gamma(X, x)$ be a positive operator as above.

Definition 3 To each subformula $\Delta(X, x, \mathbf{y})$ (where $\mathbf{y}$ is a list of variables) we associate a family of domains $\{D_{\Delta,f,\mathbf{g}}\}_{f,\mathbf{g}\in(\mathbb{N}_\perp\to\mathbb{N}_\perp)^{n+1}}$ as the minimal solution to the following:

i) $\Delta(X, x, \mathbf{y}) = t(x, \mathbf{y}) \in X$.
Let $D_{\Delta,f,\mathbf{g}} = D_{\Gamma,t(f,\mathbf{g})}$.

ii) $\Delta(X, x, \mathbf{y}) = P(x, \mathbf{y})$ (Since we use primitive recursion both for predicates and terms, it is sufficient to consider this simple case.) Let

$D_{\Delta,f,\mathbf{g}} = \mathbb{N}_\perp \to \mathbb{N}_\perp$ if $P(f, \mathbf{g})$ holds.

$D_{\Delta,f,\mathbf{g}} = (\mathbb{N}_\perp \to \mathbb{N}_\perp) \to \mathbb{N}_\perp$ if $P(f, \mathbf{g})$ does not hold.

$D_{\Delta,f,\mathbf{g}} = \{\perp\}$ if $f, \mathbf{g}$ are partial and does not contain enough information to decide $P(f, \mathbf{g})$.

iii) If $\Delta(X, x, \mathbf{y}) = \Delta_1(X, x, \mathbf{y}) \wedge \Delta_2(X, x, \mathbf{y})$,
let $D_{\Delta,f,\mathbf{g}} = \mathbb{O}_\perp \oplus (D_{\Delta_1,f,\mathbf{g}} \times D_{\Delta_2,f,\mathbf{g}})$.

iv) If $\Delta(X, x, \mathbf{y}) = \neg\Delta_1(X, x, \mathbf{y})$,
let $D_{\Delta,f,\mathbf{g}} = D_{\Delta_1,f,\mathbf{g}} \to \mathbb{N}_\perp$.

v) If $\Delta(X, x, \mathbf{y})$ is $\forall z \Delta_1(X, x, z, \mathbf{y})$,
let $D_{\Delta,f,\mathbf{g}} = \mathbb{O}_\perp \oplus \Pi(h \in \mathbb{N}_\perp \to \mathbb{N}_\perp) D_{\Delta_1,f,h,\mathbf{g}}$.

Lemma 5 a) *Up to isomorphism, there is a unique minimal solution to the domain equations of this definition.*

b) *The solution of a) will be within some ranked* $\Pi\Sigma$*-system.*

Proof
Easy, following standard methods, e.g. as described in Kristiansen and Normann [10].

The total elements in $D_{\Delta,f,\mathbf{g}}$ are defined inductively. Since X is assumed to be positive in Γ, X will have a uniqe signature in each subformula of Γ.

Definition 4 By recursion on the ordinal α and subrecursion on the subformula Δ of Γ we define the total elements $T^\alpha_{\Delta,f,\mathbf{g}}$ of $D_{\Delta,f,\mathbf{g}}$ at level α as follows:
If $\Delta(x, \mathbf{y}) = P(x, \mathbf{y})$, we let

$T^\alpha_{\Delta,f,\mathbf{g}} = (\mathbb{N}_\perp \to \mathbb{N}_\perp)_{\mathrm{tot}}$ if $P(f, \mathbf{g})$ holds.

$T^\alpha_{\Delta,f,\mathbf{g}} = ((\mathbb{N}_\perp \to \mathbb{N}_\perp) \to \mathbb{N}_\perp)_{\mathrm{tot}}$ if $P(f, \mathbf{g})$ does not hold.

If $\Delta(x, \mathbf{y})$ is of the form $t(x, \mathbf{y}) \in X$ we let $T^0_{\Delta,f,\mathbf{g}} = \emptyset$ and we let
$T^\alpha_{\Delta,f,\mathbf{g}} = \bigcup_{\beta<\alpha} T^\beta_{\Gamma,t(f,\mathbf{g})}$ for $\alpha > 0$.
For other Δ we define , $T^\alpha_{\Delta,f,\mathbf{g}}$ in the obvious way from the total elements at level α of the immediate subformulas.

The standard monotonicity properties will be satisfied:

Lemma 6 *If X occurs positivly in $\Delta(x, \mathbf{y})$, then $T^\alpha_{\Delta,f,\mathbf{g}}$ will increase with increasing α.*
If X occurs negativly in $\Delta(x, \mathbf{y})$, then $T^\alpha_{\Delta,f,\mathbf{g}}$ will decrease with increasing α.

The proof is easy and is left for the reader.

We now observe that we may define $\beta_{\delta,f,\mathbf{g}}$ in $D_{\Delta,f,\mathbf{g}}$ using a fix-point of an operator based on constructons 1.- 3. with the obvious modification and extension to cartesian products.

Lemma 7 *For each Δ, f and $\mathbf{g}$ we have that $\beta_{\Delta,f,\mathbf{g}}$ is weakly total, but not a density witness.*

Proof

We will use that we operate in a ranked $\Pi\Sigma$-system, and by induction on the rank of a compact or a separable pair we show that $\beta_{\Delta,f,\mathbf{g}}$ differs from the corresponding density witness and that the corresponding totality witness terminates on $\beta_{\Delta,f,\mathbf{g}}$. We leave the details for the reader.

Γ will define a positive operator $\hat{\Gamma}$ on Baire-space $\mathbb{N} \to \mathbb{N}$, and it is the fixpoint of this operator that we aim to represent in this section. In order to prove that we have actually constructed a representation for this set, we need a notation for the local levels in this inductively defined set. We let $\hat{\Delta}$ denote the operator defined from Δ

Definition 5 Let Γ be as above.
$A_0 = \emptyset$.
$A_\alpha = \hat{\Gamma}(\bigcup_{\beta<\alpha} A_\beta)$ for $\alpha > 0$.
We let $A_{\Delta,\alpha} = \hat{\Delta}(\bigcup_{\beta<\alpha} A_\beta)$ for $\alpha > 0$.
If X does not occur in Δ, we let $A_{\Delta,\alpha}$ be the set defined from Δ for all $\alpha \geq 0$.

Lemma 8 **a)** *If X is positive in $\Delta(x, \mathbf{y})$ and $(f, \mathbf{g}) \in A_{\Delta,\alpha}$ then $\beta_{\Delta,f,\mathbf{g}} \in T^\alpha_{\Delta,f,\mathbf{g}}$.*

b) *If X is negative in $\Delta(x, \mathbf{y})$ and $(f, \mathbf{g}) \notin A_{\Delta,\alpha}$, then $\beta_{\Delta,f,\mathbf{g}}$ can be separated from any $y \in T^\alpha_{\Delta,f,\mathbf{g}}$ by a totality witness.*

Proof

We use induction on α and subinduction on Δ.
For atomic statements $P(x, \mathbf{y})$ the properties of both a) and b) will hold, and it follows from the proof og Lemmas 2 and 3 that the properties of both a) and b) will hold for all Δ where X does not occur.
Thus we may assume that the appropriate a) or b) will hold for all immediate subformulas of a formula.

Letting $A_{\Gamma,0} = A_0$ we see that the lemma trivially holds for $A_{\Gamma,0}$.
The case $t(x, \mathbf{y}) \in X$ for $\alpha > 0$ will follow from the induction hypothesis on $A_{\Gamma,\beta}$ for $\beta < \alpha$.
The rest of the cases will follow locally for one α by induction on Δ, using essentially Lemmas 2 and 3.
This ends the proof.

Lemma 9 **a)** *If X is positive in $\Delta(x, \mathbf{y})$ and $\beta_{\Delta,f,\mathbf{g}}$ is inseparable from a $y \in T^{\alpha}_{\Delta,f,\mathbf{g}}$, then $(f, \mathbf{g}) \in A_{\Delta,\alpha}$.*

b) *If X is negative in $\Delta(x, \mathbf{y})$ and $\beta_{\Delta,f,\mathbf{g}} \notin T^{\alpha}_{\Delta,f,\mathbf{g}}$, then $(f, \mathbf{g}) \notin A_{\Delta,\alpha}$.*

Proof
We use induction on α and subinduction on Δ.
As in the previous proof, a) and b) will hold for Δ where X does not occur in Δ. So we may assume that the appropriate a) or b) will hold for any immediate subformula of a formula Δ.
In the case $t(x, \mathbf{y}) \in X$ we see that the assumption in a) is vacous for $\alpha = 0$, and a) holds by the induction hypothesis on $T^{\beta}_{\Gamma,t(f,\mathbf{g})}$ for some $\beta < \alpha$ for $\alpha > 0$.
The case $\Delta = \Delta_1 \wedge \Delta_2$ is easy, and for the case $\Delta = \forall z \Delta_1$ we can use the methods of the proof of Lemma 3.
So we concentrate on the case $\Delta = \neg\Delta_1$.
Proof of a): Assume that X is positive in Δ and that $\beta_{\Delta,f,\mathbf{g}}$ is separable from $y \in T^{\alpha}_{\Delta,f,\mathbf{g}}$.
Now $D_{\Delta,f,\mathbf{g}} = D_{\Delta_1,f,\mathbf{g}} \to \mathbb{N}_{\perp}$ and (omitting the index on h)

$$\beta_{\Delta,f,\mathbf{g}} = \lambda z.\mu k(h(\beta_{\Delta_1,f,\mathbf{g}})(k) \neq h(z)(k)).$$

If $y(\beta_{\Delta_1,f,\mathbf{g}}) = n$, let z be a density witness such that $y(z) = n$ and $h(z)(k) = h(\beta_{\Delta_1,f,\mathbf{g}})(k)$ for all $k \leq n + 1$. Since y and $\beta_{\Delta,f,\mathbf{g}}$ are inseparable, we have $y(z) = \beta_{\Delta,f,\mathbf{g}}(z)$. But

$$n + 1 \leq \beta_{\Delta,f,\mathbf{g}}(z) = y(z) = y(\beta_{\Delta_1,f,\mathbf{g}}) = n,$$

a contradiction. Thus $y(\beta_{\Delta_1,f,\mathbf{g}})$ is undefined.
Since $y \in T^{\alpha}_{\Delta,f,\mathbf{g}}$ we must have that $\beta_{\Delta_1,f,\mathbf{g}} \notin T^{\alpha}_{\Delta_1,f,\mathbf{g}}$, and by the induction hypothesis $(f, \mathbf{g}) \notin A_{\Delta_1,\alpha}$. But then $(f, \mathbf{g}) \in A_{\Delta,\alpha}$.

Proof of b): Assume that X is negative in Δ and assume that $\beta_{\Delta,f,\mathbf{g}} \not\in T^{\alpha}_{\Delta,f,\mathbf{g}}$. Then for some $z \in T^{\alpha}_{\Delta_1,f,\mathbf{g}}$ we have that $\beta_{\Delta,f,\mathbf{g}}(z)$ is undefined. This must be because z is inseparable from $\beta_{\Delta_1,f,\mathbf{g}}$.
By the induction hypothesis $(f,\mathbf{g}) \in A_{\Delta_1,\alpha}$, so $(f,\mathbf{g}) \not\in A_{\Delta,\alpha}$.

We leave the rest of the details for the reader.

We then have proved the following theorem by direct construction

Theorem 1 *Let $\Gamma(X,x)$ be a formula where the set variable X occurs positivly, defining a monotone operator $\hat{\Gamma}$ on $\mathbb{N} \to \mathbb{N}$. Then there is a full representation of the least fixpoint of $\hat{\Gamma}$.*

Remark It is not just the simple fact that there is a representation of this set that is of importance in this result, but the principles of type construction whose interpretation is used in the construction of the representation. In this case we have used one inductively defined type to find the representation, and the argument shows that the complexity of a domain theoretic interpretation of a type defined by positive induction is as high as that of the closure of an arbitrary positive inductive definition over Baire space. In the next section we will estimate the complexity of the domains with totality defined by iterated positive induction in Kristiansen and Normann [10]. If Y is an extra set variable and X is positive in $\Gamma(x,X,Y)$, we can construct a full representation for $\mu X(X = \{x \mid \Gamma(x,X,Y)\})$ uniformly in a full representation for Y. We simply use the representation for Y as $D_{\Delta,f,\mathbf{g}}$ and $\beta_{\Delta,f,\mathbf{g}}$ if $\Delta(x,\mathbf{y})$ is $t(x,\mathbf{y}) \in Y$.
This gives us the following

Corollary 1 *If A is positive inductively definable in B, we can construct a full representation of A uniformly in a full representation for B.*

4 Representing transfinite computations

In this section we will combine the methods from Normann [14] and the previous section to prove a representation theorem for computations relative to the functional J defined below. Transfinite computations are defined as in Kleene [5].
First we let E_1 be the *partial* functional of type 3 defined by

$E_1(F) = 0$ if $F(f) = 0$ for all f

$E_1(F) = 1$ if $F(f) = 1$ for some f

(accepting F to be partial).

We then define the type three functional J only on total F by

$J(F, e) = 0$ if $\{e\}(E_1, F) = 0$

$J(F) = 1$ otherwise.

The idea is that for each application of J we copy the full representation developed in Section 3 as a local part of the representation we develope for the whole relation.

Lemma 10 *Let F be a total functional of type 2, $\mathbf{f}$ a finite sequence of natural numbers and functions in $\mathbb{N}^{\mathbb{N}}$.*
The relation $\{e\}(E_1, F, \mathbf{f}) = n$ is defined by a positive induction uniformly in the graph of F.

The proof is trivial.

Definition 6 A *representation* of a functional F will be a full representation of its graph.

Remark In the main theorem of Normann [14] we showed that if $\{e\}({}^3E, \mathbf{f})\downarrow$ then uniformly in $\mathbf{f}$ we have a full representation of the relation $\{e\}({}^3E, \mathbf{f}) = n$ indexed by the natural numbers.

Before we can prove the final representation theorem for computations relative to the functional J we need to select the domain of parameterisation with some care. It is possible to use the domain T with the total objects T_{IND} from Kristiansen and Normann [10], but the problem is that not every inductively defined domain will satisfy density, and even the set of indices in T for the domains for which the set of total objects are defined is not dense in T. It is possible to overcome these problems working strictly within the hierarchy of [10]. We will, however use the flexible concept of a $\Pi\Sigma$-system from Berger [3].

Definition 7 We define the domain T and the subdomain S together with the interpretation $I(t)$ as a domain for each $t \in T$ as the minimal solution to the following set of domain equations:

i) The base type domains $\mathbb{O}_\perp$, $\mathbb{B}_\perp$ and $\mathbb{N}_\perp$ are represented in S by atomic elements O, B and N and with the respective domains as their interpretations.

ii) If $t \in S$ and $F : I(t) \to S$, then $(\Pi, t, F) \in S$ with $I(\Pi, t, F) = \Pi(x \in I(t))I(F(x))$.

iii) If $F : ((\mathbb{N}_\perp \to \mathbb{N}_\perp) \times \mathbb{N}_\perp) \to S$ is continuous, if $\Gamma(X)$ is the formula inductively defining computations in E_1 relative to a functional with graph X, if $\Delta(\mathbf{x})$ is a subformula of Γ and if $\mathbf{f} \in (\mathbb{N}_\perp \to \mathbb{N}_\perp)^n$ then $(\Delta, \mathbf{f}, F) \in T$. $I(\Delta, \mathbf{f}, F)$ will correspond to $D_{\delta,\mathbf{f}}(F)$ of the proof of Corollary 1, where we assume that F is the parameterisation in a full representation of X.
We let $(\Gamma, f, F) \in S$ for all $f \in (\mathbb{N}_\perp \to \mathbb{N}_\perp)$.

We now define the hierarchies S_{wf} and T_{wf} of well founded objects in S and T respectivly. Simultanously we define the total elements of $I(t)$ for $t \in T_{\mathrm{wf}}$.

Definition 8 **i)** O, B and N are in S_{wf} with the obvious interpretation of the total elements.

ii) If $t \in S_{\mathrm{wf}}$ and $F : I(t)_{\mathrm{tot}} \to S_{\mathrm{wf}}$ then $(\Pi, t, F) \in S_{\mathrm{wf}}$ with the obvious interpretation of totality.

iii) If $F : (\mathbb{N} \to \mathbb{N}) \to S_{\mathrm{wf}}$ and Δ, $\mathbf{f}$ are as in Definition 7, then $(\Gamma, f, F) \in S_{\mathrm{wf}}$ with the total objects as defined in section 2.
Moreover, all relevant $(\Delta, \mathbf{f}, F) \in T_{\mathrm{wf}}$ with total objects as defined in section 2.

We consider S_{wf} as a subset of T_{wf}.

Lemma 11 **a)** *The parameterisation* $T, \{I(t)\}_{t \in T}$ *with* T_{wf} *and* $I(t)_{\mathrm{tot}}$ *is contained in a ranked* $\Pi\Sigma$*-system.*

b) *Uniformly in* $t \in T_{\mathrm{wf}}$ *we can find a total object in* $I(t)$*.*

Proof
The proof of a) is trivial. In order to prove b) we combine the recursion theorem connected with the inductive definition of S_{wf} with the proof of the analogue statement for the representation for inductively defined sets from

Section 2.

The parameterisation $\{I(t)_{\text{tot}}\}_{t\in T_{\text{wf}}}$ will satisfy Berger's concepts of density and totality, see [3].

We now prove density of S_{wf} in S and of T_{wf} in T.

Lemma 12 *If τ is a compact element of T there is an extension of τ to a $t \in T_{\text{wf}}$. Moreover, if $\tau \in S$, then $t \in S_{\text{wf}}$.*

Proof
We use induction on the rank of τ and only two cases are nontrivial.
Case 1: $\tau = (\Pi, \tau_1, F_0) \in S$.
Then τ_1 is compact in S and by the induction hypothesis τ_1 can be extended to a $t_1 \in S_{\text{wf}}$. We then use totality in $I(t_1)$ and the density of S_{wf} obtained from the induction hypothesis to extend F_0 to an $F : I(s) \to S$ such that $(\Pi, t_1, F) \in S_{\text{wf}}$.
Case 2: $\tau = (\Delta, \mathbf{f}_0, F_0)$.
It is then trivial to extend $\mathbf{f}_0$ to a sequence $\mathbf{f}$ of total functions, and we use the density of S_{wf} in S obtained from the induction hypothesis to extend F_0 to a total F. Then $(\Delta, \mathbf{f}, F) \in T_{\text{wf}}$, and we are through.

We will not need that T_{wf} satisfies co-density, but for the sake of completeness we state it anyhow. This lemma can be seen as a special case of co-density for the universes in Berger [4]

Lemma 13 *Let σ_1 and σ_2 be inconsistent compacts in T. Then σ_1 and σ_2 can be separated by a Boolean-valued function total on T_{wf}.*

We are now ready to prove our main representation theorem. Since we are considering computations relative to the functional J, we simplify the notation by writing $\{e\}^J(\mathbf{f})$ for computations in J and $\mathbf{f}$. We then don't have to bother about where in the input list J occurs.

Theorem 2 *There is a representation of the relation $\{e\}^J(\mathbf{f}) = n$ into S_{wf}, $\{I(s)_{\text{tot}}\}_{s\in S_{\text{wf}}}$.*

Proof
By the fix-point theorem for domains we define $\phi(e, \mathbf{f}, n) \in S$ and $\beta(e, \mathbf{f}, n) \in I(\phi(e, \mathbf{f}, n))$ following our standard constructions of representation. We will show that when $\{e\}^J(\mathbf{f})\downarrow$, then $\phi(e, \mathbf{f}, n) \in S_{\text{wf}}$, and in this case $\{e\}^J(\mathbf{f}) = n$ if and only if $\beta(e, \mathbf{f}, n)$ is total.
We do not give the construction in detail, but focus on the important aspects:

* If e is an index for an initial computation (S1, S2, S3 or S7) or not an index for any of the eight schemes we consider, we treat the statement $\{e\}^J(\mathbf{f}) = n$ as an atomic statement that is immediatly true or false.

* Composition
$$\{e\}^J(\mathbf{f}) = \{e_1\}^J(\{e_2\}^J(\mathbf{f}), \mathbf{f})$$
is handled as
$$\{e_1\}^J(\mathbf{f}) = n \Leftrightarrow \forall m(\{e_2\}^J(\mathbf{f}) = m \rightarrow \{e_1\}^J(m, \mathbf{f}) = n)$$
viewing $\forall m(\{e_2\}^J\mathbf{f} = m \rightarrow \ldots)$ as a bounded quantifier.

* In the cases of permutation (S6) and diagonalisation (S9) we use the values of ϕ and β directly.

* In the application of J, i.e. S8, we define the computations in E_1 relative to a function F via a positive inductive definition relative to the graph of F.
If $F = \lambda g\{e_1\}^J(g, \mathbf{f})$, then, by the proper induction hypothesis, $\lambda(g, n)\phi(e_1, g, \mathbf{f}, n)$ and $\lambda(g, n)\beta(e_1, g, \mathbf{f}, n)$ will be a strong representation of the graph of F, and we will use that in our further constructions. Precisly we have to define $\phi(e, d, \mathbf{f}, n)$ and $\beta(e, d, \mathbf{f}, n)$ when $\{e\}^J(d, \mathbf{f}) = J(\lambda g.\{e_1\}^J(g, \mathbf{f}), d)$.
From analysing the inductive definition we get an object (Γ, d, F) and a $\gamma(d, F)$ such that $\gamma(d, F)$ is total in $I(\Gamma, d, F)$ if and only if $\{d\}(E_1, F) = 0$. We will use this to represent $\{e\}^J(d, \mathbf{f}) = 0$, the representation of the negation to represent $\{e\}^J(d, \mathbf{f}) = 1$ and the representation of immediate falsity to represent $\{e\}^J(d, \mathbf{f}) = n$ for $n \neq 0, 1$.

Since the representation is constructed as a fix-point we authomatically get approximations by compacts. These approximations are used uniformly in the case of composition. Since the enumeration of the approximation is only used positivly, this is no obstacle to the validity of the construction.
It is now easy to show by induction on the length of the computation that this representation works.

An even sharper result would have been the following:

In addition we have for all n that $\{e\}^J(\mathbf{f})\downarrow$ if and only $\phi(e, \mathbf{f}, n) \in S_{\mathrm{wf}}$.

The theorem can be improved in this way, but we will not give the proof here.

Corollary 2 *The complexity of the hierarchy* T_{IND}*,* $\{I(t)_{\mathrm{tot}}\}_{t \in T_{\mathrm{IND}}}$ *from Kristiansen and Normann [10] coincides with semicomputability in* J *relative to arbitrary* $f \in \mathbb{N}^{\mathbb{N}}$*.*

Proof
It is easy to embed our hierarchy S_{wf} into T_{IND} preserving the interpretation and the set of total objects. This will show that the hierarchy from [10] is at least as complex as semicomputability in J.
Conversly it is easy to see that T_{IND} is definable in $L_\kappa(\mathbb{N}^{\mathbb{N}})$ where κ is the least ordinal where $L_\kappa(\mathbb{N}^{\mathbb{N}})$ is admissible and a limit of admissibles. This structure is again known as the *companion* of J, and shares the complexity of semirecursion in J.

5 Domain-representation of $L_\kappa(HC)$

5.1 Representation of structures in general

So far we have defined representation of a set as reducing elementhood to totality. In connection with semicomputability this is a natural approach. Now we will discuss representations of relations. It is then natural to assume that we have representations of both the positive and the negative part of the relation. This leads us to:

Definition 9 a) A *representation of a relation on the total elements in a domain* will consist of full representations of the set of elements of the relation and its complement.

b) Let $\mathcal{A}$ be a first order structure without functions (a relational structure) where the ground set is the total elements of some domain. A *representation* of $\mathcal{A}$ will be a set of representations for each relation in $\mathcal{A}$.

c) Let $\mathcal{A} = (A, R_1, \ldots, R_n)$ be a relational structure.
A *pre-representation* of $\mathcal{A}$ will be a domain X with totality $\bar{X}$ and relations $I, P_1, \ldots, P_n$ on $\bar{X}$ where

i) I is a binary equivalence relation (I for 'identity').

ii) For each i, P_i will respect I and have the same arity as R_i,

together with a representation of $I, P_1, \ldots, P_n$ and together with an isomorphism between $\mathcal{A}$ and $(\bar{X}, P_1, \ldots, P_n)/I$.

We are going to show how we can find representations of certain ordinals κ, and in those cases, how we can find representations of the structures $L_\kappa(HC)$. First we will see how we can construct representations of HC, the starting point of our general construction. For the rest of this subsection we will let X be the minimal solution to the domain equation

$$X = \mathbb{O}_\perp \oplus (\mathbb{N}_\perp \to X).$$

We will use the letters l and r for the *inleft* and *inright* operators into the disjoint sum.

The total objects $\bar{X}$ are defined inductively by

$l(0) \in \bar{X}$

If $r(f) \in X$ and $f(n) \in \bar{X}$ for all $n \in \mathbb{N}$, then $r(f) \in \bar{X}$.

$\bar{X}$ will simply be the set of well founded trees with branching over $\mathbb{N}$ and with 0 at the leaves.

Let $\rho : \bar{X} \to HC$ be defined as follows:

$\rho(l(0)) = \emptyset$

$\rho(r(f)) = \{\rho(f(n)) \mid n \in \mathbb{N}\}$.

Then ρ maps $\bar{X}$ onto HC.
If we now define the relation

$I(x, y) \Leftrightarrow \rho(x) = \rho(y)$

$E(x, y) \Leftrightarrow \rho(x) \in \rho(y)$

on $\bar{X}$, we see that a representation of $(\bar{X}, E, I)$ will be a pre-representation of HC.

There are several ways to construct a representation of this sort, we will indicate three constructions not giving the details of any of them.

First we observe that I is inductively defined and that E is directly definable from I. The first construction is based on the general methods for handling inductive definitions. The disadvantage of this method is that the complexity of the domains with totality used in the representations goes far beyond what is needed, since we replace occurences of $\vee$ and $\exists$ by $\neg$, $\wedge$ and $\forall$, and it is in particular the representation of $\neg$ that increases the complexity.

Our second construction is based on the observation that the inductive definition of I essentially is an inductive definition on a discrete structure, the set of predecessors of x and y resp.

Definition 10 Let $x \in X$. We define the tree $T(x)$ by recursion on x as a tree of sequences of natural numbers as follows:
If $x = l(0)$, then $T(x)$ will consist of the empty sequence as its only node.
If $x = r(f)$ then $T(x)$ will consist of the empty sequence together with all sequences of the form $n * \sigma$ for $\sigma \in T(f(n))$.

We seemingly only defined $T(x)$ for $x \in \bar{X}$, but T can be defined up to isomorphism as the total elements in the minimal solution of the equations

$$T(l(0)) = \mathbb{O}_\perp$$

$$T(r(f)) = \mathbb{O}_\perp \oplus \Sigma(n \in \mathbb{N})T(f(n))$$

We then have

Lemma 14 *The relation $I(x,y)$ can be expressed by a formula*

$$I(x,y) \Leftrightarrow \forall g \in (T(x) \times T(y) \to \mathbb{N})R(g)$$

where R is expressed using quantifiers over $\mathbb{N}$, the concatenation operator $*$, the constant for the empty sequence and the x-dependent predicate isolating the end-nodes of the tree $T(x)$.

The proof is standard.
We now observe that the standard method for eliminating number quantifiers when a universal function quantifier is present can be used in this setting too, that the complement of I is also inductively definable and that the representations for E and $\neg E$ can be constructed directly from representations for I and $\neg I$ resp. Thus all relations I, $\neg I$, E and $\neg E$ can be defined by a formula of the form $\forall g \in D(x,y)\exists n R'(g,n)$ where D has the complexity of type 1

and R' is decidable. We can then easily construct a representation.
This ends our discussion of the second construction.

The third construction will be by using $Ct(2)$ as the domain where totality represents I and E both positivly and negativly. This will require some standard coding and is essentially equivalent to the second construction.
We end this section by stating the conclusion of this third construction without giving any further details.

Theorem 3 *Let $(X, \bar{X})$ be the domain with totality of well founded trees with branchings over $\mathbb{N}$ as defined above. Let $\rho : \bar{X} \to HC$ be the canonical map of these well founded trees onto HC. Let I and E be defined as in the text.*

Then there are representations ϕ_1, ϕ_2 , ϕ_3 and ϕ_4 of I, $\neg I$, E and $\neg E$ into $(\mathbb{N}_\perp \to \mathbb{N}_\perp) \to \mathbb{N}_\perp$.

5.2 Representing ordinals

In this section we will see how we can obtain pre-representations of ordinals from well founded relations that will appear in our hierarchies of domains with totality. Our methods will be general, and we do not make any attempt to keep the complexity of the representations down.

As a digression we observe that the domain $(X, \bar{X})$ used in the previous section can be used as the base of a pre-representation of $\aleph_1$, and that we can use totality in type 2 as the representing property. There will be other cases as well where special methods give representations that are far simpler in complexity than those obtained by the general methods developed here. We will not go into further details.

In general we will look for prewellorderings $(\bar{X}, \leq)$ with a positive representation of $\leq$. Since we can construct representations for any boolean combination, we will in constructions assume that we also have representations for the induced equivalence and for the strict ordering. This will in turn give us a pre-representation of an ordinal.

Before we see how we can find representations for some large ordinals, let us see how to construct representations for the successor and for the least upper bound of the lengths of some parameterisation of representations.

5.2.1 The successor

Let $(\bar{X}, \leq)$ be a representable prewellordering of ordinal hight α. Let the representation be

$$x \leq y \leftrightarrow \phi(x, y) \in \bar{A}_{x,y}$$

Let $Y = X \oplus \mathbb{O}_\perp$.
Let $\psi(l(x), l(y)) = \phi(x, y) \in A_{x,y}$.
Let $\psi(l(x), r(0)) = \psi(r(0), r(0)) = \lambda n.n \in \mathbb{N}_\perp \to \mathbb{N}_\perp$.
Let $\psi(r(0), l(x)) = \lambda f.\mu n(f(n) \neq n) \in (\mathbb{N} \to \mathbb{N}_\perp) \to \mathbb{N}_\perp$.

We just constructed a representable prewellordering of hight $\alpha + 1$

5.2.2 Union

When we are going to represent properties defined by more general cases of induction than those considered so far, it will be an advantage to be able to isolate where the induction starts. For this purpose we will from now on restrict ourselves to a set of domains with prewellorderings on the total elements where the minimal elements can be isolated.

Definition 11 **a)** A representable prewellordering $(\bar{X}, \leq)$ has *identifiable* 0 if the equivalence class of minimal objects is a closed and open subset of $\bar{X}$.

b) A dependent family $\{(\bar{Y}_x, \leq_x)\}_{x \in \bar{X}}$ of uniformly representable prewellorderings has *uniformly identifiable* 0's if each $(\bar{Y}_x, \leq_x)$ has identifiable 0 and the set of minimal objects is uniformly closed and open in $\bar{Y}_x$.

We will only work with representations with identifiable 0, and we will let $x = 0$ simply mean that x is minimal in the prewellordering at hand.
It is clear that our construction of the successor preserves this property uniformly.

Now let $\{Y_x\}_{x \in X}$ be a parameterised family with totality $\{\bar{Y}_x\}_{x \in \bar{X}}$, with uniformly representable prewellorderings $\leq_x$ which again have uniformly identifiable 0's.
We will define a prewellordering on $\bar{Z} = \Sigma(x \in \bar{X})\bar{Y}_x$ and show how it can be represented.

If we let $\|y\|_x$ be the ordinal rank of y in $\leq_x$ we want

$$(x,y) \leq (x',y') \Leftrightarrow \|y\|_x \leq \|y'\|_{x'}$$

Now this relation can be inductively defined as follows

$$(x,y) \leq (x',y') \Leftrightarrow \forall z <_x y \exists z' <_{x'} y'(z,y) \leq (z',y')$$

or, if we replace $\exists$ by $\neg$ and $\forall$, we get

$$(x,y) \leq (x',y') \Leftrightarrow (\forall z <_x y)\neg(\forall z' <_{x'} y')\neg((z,y) \leq (z',y')).$$

This is easily seen to have identifiable 0.
In order to construct a well formed representation we observe that the inequality is trivially correct if $y =_x 0$ and trivially false if $y \neq_x 0$ but $y' =_{x'} 0$.

In the *otherwise*-case we use the general method of representing bounded quantification.

Now we define the domain $B_{(x,y),(x',y')}$ that we will use in the representation of this new prewellordering.
$B_{(x,y),(x',y')} = \mathbb{N}_\perp \to \mathbb{N}_\perp$ if $y =_x 0$
$B_{(x,y),(x',y')} = (\mathbb{N}_\perp \to \mathbb{N}_\perp) \to \mathbb{N}_\perp$ if $y' =_{x'} 0$ and $y \neq_x 0$.
In the case where both are different from 0 we let
$B_{(x,y),(x',y')} = \Pi(z \in Y_x)\Pi(u \in A_{z,x})(\Pi(z' \in Y_{x'})\Pi(v \in A_{z',x'})(B_{(x,z)(x',z')} \to \mathbb{N}_\perp)(mod\ v) \to \mathbb{N}_\perp)(mod\ u)$.

The representing function will suggest itself. For the induction start we use basic truth or basic falsity and for the induction step we use the standard constructions of representing functions.
In order to complete the construction we need to specify the k'th approximation.

> The k'th approximation to $\mathbb{N}_\perp \to \mathbb{N}_\perp$ is itself, and so is the k'th approximation to $\lambda n.n$
>
> The k'th approximation to $(\mathbb{N}_\perp \to \mathbb{N}_\perp) \to \mathbb{N}_\perp$ is itself, and the k'th approximation to $\lambda f.\mu n(f(n) \neq n)$ is $\lambda f.\mu n(n = k \vee f(n) \neq n)$ (which is total).
>
> In the otherwise case we let $B^0_{(x,y),(x',y')} = \mathbb{N}_\perp \to \mathbb{N}_\perp$ and the 0'th approximation to the representing function will be $\lambda n.n$.

The induction step will be
$B^{k+1}_{(x,y),(x',y')} = \Pi(z \in Y_x)\Pi(u \in A_{z,x})(\Pi(z' \in Y_{x'})\Pi(v \in A_{z',x'})$
$(B^k_{(x,z)(x',z')} \to \mathbb{N}_\perp)(mod\ v) \to \mathbb{N}_\perp)(mod\ u)$, where of course B^{k_1} is the k_1'th approximation to B^k when $k_1 \leq k$.

By induction on $\max\{\|y\|_x, \|y'\|_{x'}\}$ we see that $B_{(x,y),(x',y')}$ is constructed from the given domains with totality by a well-founded use of dependent products, and that they thus themselves are well defined as domains with totality. We have that B will be the limit of the B^k's in the domain-theoretical sense. This is essential, because otherwise our construction would not be continuous. Still we omit the details here.

Finally we have to show that this construction really leads to a representation of the defined prewellordering. This is proved by a simple induction on the rank $\|y'\|_{x'}$. When one of y and y' is a 0-element our construction is directly designed to produce a representation, and otherwise we use the induction hypothesis and the general properties of our operations on representations.

Our constructions are uniform in the given data, so we have actually proved the following result:

Theorem 4 *Let $\{\bar{Y}_x\}_{x \in \bar{X}}$ be a parameterisation of domains with totality, and let $\{\leq_x\}_{x \in \bar{X}}$ be a family of prewellorderings with identifiable 0's via the subsets $\{O_x\}_{x \in \bar{X}}$ that are both closed and open, and with a uniform representation $\phi(z, y, x)$ into $\bar{A}_{z,y,x}$ of the relation $z \leq_x y$.Then there is a continuous operator $\bigsqcup$ such that $\bigsqcup(\{\bar{Y}_x\}_{x \in \bar{X}}, \phi, \{\bar{A}_{z,y,x}\}_{x \in \bar{X},(y,z) \in (\bar{Y}_x)^2}, \{O_x\}_{x \in \bar{X}})$ is a representation of a prewellordering with identifiable 0's whose ordinal hight is the supremum of the ordinal heights of each individual $\leq_x$.*

In the case where $\bar{X}$ itself has a representable prewellordering we may also construct the sum of the prewellorderings $\leq_x$. In paricular, if $\bar{X}$ has a representable wellordering this construction will be far simpler than the construction above. This will be the case when X is the domain of boolean values or the domain of natural numbers. Every computable ordinal can of course be represented by a discrete domain.

5.2.3 Representing some definable ordinals

We will now see how we can obtain representations for ordinals obtained via computations in J.

Lemma 15 *Let C_J be the set of J-computations $< e, \mathbf{f}, n >$ where $\{e\}^J(\mathbf{f}) \approx n$.*
The elements of C_J are normed by $\|\sigma\|_J =$ the ordinal rank of the computation σ, seting $\|\sigma\|_J = \infty$ if $\sigma \notin C_J$.
There is a partial J-computable function sc (stage comparison) such that if $\sigma \in C_J$, then $sc(\sigma, \tau)\downarrow$ for all τ and

$$sc(\sigma, \tau) = 0 \Leftrightarrow \|\tau\|_J \leq \|\sigma\|_J$$

The proof is standard and will be omitted.

Corollary 3 *If $\alpha = \|\sigma\|$ is an ordinal, then continuously in σ we can find a pre-representation of an ordinal larger that α.*

5.3 Representation of $L_\kappa(HC)$

Theorem 5 *Uniformly in a pre-representation for the ordinal κ there is a pre-representation for $(L_\kappa(HC), \in)$.*

Proof
We will not introduce any new methods here, and we will be brief in describing the construction.

The main step will be a uniform construction of a representation of $L_{\kappa+1}(HC)$ from a representation of $L_\kappa(HC)$.
Assuming that this is done, let us scetch the global construction. So, let $(\bar{X}, \leq)$ be a prewellordering with a representation and with identifiable 0's. For each $x \in \bar{X}$ we construct a representation for $L_{\|x\|}(HC)$ based on the following definition

$L_0(HC) = HC$, for which we have constructed a representation.

$L_\kappa(HC) = \bigcup_{\kappa_1 < \kappa} L_{\kappa_1+1}(HC)$ for $\kappa > 0$.

We use the same method as for ordinals to handle the union.

Now assume that $(\bar{X}, E, I)$ is a representable structure isomorphic to $L_\kappa(HC)$ when divided out by the equivalence relation I. We construct a pre-representation for $L_{\kappa+1}(HC)$ in five steps:

Step 1 Uniformly in a Gödel-number e we construct a representation for each element in $\mathcal{D}_{\omega,n,\kappa}$ as subsets of $\bar{X}^n$, where $\mathcal{D}_{\omega,n,\kappa}$ is the set of definable subsets of $L_\kappa(HC))^n$.

Step 2 If e is the Gödel-number of $A \in \mathcal{D}_{\omega,n+1,\kappa}$ and $x_1, \ldots, x_n \in \bar{X}$ represent objects $u_1, \ldots, u_n \in L_\kappa(HC)$, we let $(e, x_1, \ldots, x_n)$ represent the object $A_{u_1,\ldots,u_n} = \{x \mid (x, u_1, \ldots, u_n) \in A\}$ in $L_{\kappa+1}(HC)$.

Step 3 We construct positive and negative representations for equality between sets represented by $(e, x_1, \ldots, x_n)$ and $(d, y_1, \ldots, y_m)$.
Equality is inclusion both ways, and here we have a choice between expressing inclusion via bounded or unbounded quantifiers, in the last case expressing the implication using $\neg$ and $\wedge$.

Step 4 We construct a representation of identity between an element $A_{u_1,\ldots,u_n}$ of $L_{\kappa+1}(HC)$ represented by $(e, x_1, \ldots, x_n)$ and an element v of $L_\kappa(HC)$ represented by $y \in \bar{X}$. But the representatives for the elements of $A_{u_1,\ldots,u_n}$ and for the elements of v are elements of $\bar{X}$ and they will represent the same object in the two cases. Then we may use a construction analogue to the one used in Step 3.

Step 5 Finally we will represent $\in$ in $L_{\kappa+1}(HC)$. This is now easy since
$A_{e,u_1,\ldots,u_n} \in A_{d,v_1,\ldots,v_n} \Leftrightarrow$
$(\exists w \in L_\kappa(HC))(w = A_{e,u_1,\ldots,u_n} \wedge (w, v_1, \ldots, v_n) \in A_d)$.
We use general methods for representing this statement.

This ends our proof of the theorem.

Corollary 4 *Let κ be computable in J and $f : \mathbb{N} \to \mathbb{N}$.
Uniformly in f and an index for a computation in J and f of length κ we can construct a representation for $L_\kappa(HC)$*

Of course, if we find stronger hierarchies of domains with totality, density and co-density, we will also produce representations for longer initial segments of $L(HC)$.

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