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Optimal Portfolio, Partial Information and Malliavin Calculus

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Abstract

In a market driven by Lévy processes, we consider an optimal portfolio problem for a dealer who has access to some information in general smaller than the one generated by the market events, in this sense we refer to this dealer as having *partial information*. For this generally incomplete market and within the non-Markovian setting, we give a characterization for a portfolio maximizing the expected utility of the final wealth. Techniques of Malliavin calculus are used for the analysis.

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1. Introduction

During the last decade there have been many applications of Malliavin calculus to finance in the literature. So far the applications have essentially been within one of the following two areas:

(i) Explicit formulae for the replicating portfolios or closest hedge of given claims essentially based on the Clark-Ocone formula (see e.g. [KO], [BDLØP])
(ii) Numerically tractable formulae for the computation of the "Greeks" in finance, for example the δ-hedge (see e.g. [FLLLT]).

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The purpose of this paper is to present a different type of financial application of Malliavin calculus, namely in the problem of finding an *optimal portfolio*. More precisely, we are given a utility function and we seek a portfolio (within an admissible class of portfolios) which maximizes the utility of the corresponding terminal wealth. This is a *stochastic control* problem. Traditionally there have been two approaches to this type of problem:

(a) A *dynamic programming* or *stochastic maximum principle* approach. These approaches require that the system is *Markovian*. See e.g. [YZ], and references therein, for more information about these methods.
(b) The *martingale approach* or *duality approach*. This is an efficient method if the financial market equations have a special "multiplicative" form and the market is complete. An advantage is that the system need not be Markovian. We refer to e.g. [KS], and references therein, for a presentation of the martingale method.

There are, however, many situations where the problem of finding the optimal portfolio cannot be approached by any of the two methods (a) and (b) above. For example, if the information $\mathcal{E}_t$ available to the agent at time $t \geq 0$ is strictly less than the information $\mathcal{F}_t$ that can be obtained by perfect observations of the processes driving the market prices up to time t, then the system may be neither Markovian nor complete. This is the case if, for example,

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}, \qquad t \geq 0,$$

where $\delta > 0$ is a fixed *information delay*.

In this paper we will show that Malliavin calculus can be used to solve optimal portfolio problems also in such partial information cases.

We emphasize the distinction between the *partial information* control problem studied in this paper and the *partial observation* control problem, where the control is based on noisy observations of the state. The latter problem has been studied by many authors, but the methods and results of the partial observation problem do not apply to our situation.

Our main result (Theorem 3.1) gives an explicit equation for the optimal portfolio π, if it exists. This equation relates π to the coefficients in the market equations and to the Malliavin derivatives of a certain function of the optimal terminal wealth $X_\pi(T)$.

2. Some basic definitions and results

We first recall that a *Lévy process* $\eta = \eta(t) = \eta(t,\omega)$, $(t,\omega) \in [0,\infty) \times \Omega$, on a complete probability space $(\Omega, \mathcal{F}, P)$ is a process with stationary and independent

increments and càdlàg paths (i.e. right continuous paths with left sided limits). We assume that $\eta(0) = 0$. The (possible) jump of η at time t is defined by

$$\Delta\eta(t) = \eta(t) - \eta(t^-). \tag{2.1}$$

The *jump measure* N of η is defined by

$$N\big((a,b],U\big) := \sum_{t\in(a,b]} 1_U(\Delta\eta(t)), \tag{2.2}$$

i.e. by the number of jumps of size $\Delta\eta(t) \in U$ for $t \in (a,b]$. Here $0 \le a < b \le \infty$ and $U \in \mathcal{B}(\mathbb{R}_0)$, i.e. U is a Borel set with $\bar{U} \subset \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. The differential form of this random measure is denoted by $N(dt,dz)$. The *Lévy measure* of η is defined by

$$\nu(U) := E\big[N\big((0,1],U\big)\big], \qquad U \in \mathcal{B}(\mathbb{R}_0) \tag{2.3}$$

and the *compensated jump measure* (or *compensated Poisson random measure*) of η is defined by

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt. \tag{2.4}$$

In the sequel we assume that

$$E\big[\eta^2(t)\big] < \infty, \qquad t \ge 0. \tag{2.5}$$

The *Lévy-Itô decomposition theorem* states that if (2.5) holds, then there exist constants a and b such that

$$\eta(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds,dz), \qquad t \ge 0, \tag{2.6}$$

where B is a standard Brownian motion independent of N. In view of this we see that it is natural to study processes which have the form

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\widetilde{N}(ds,dz), \tag{2.7}$$

or, in differential form,

$$dX(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\widetilde{N}(dt,dz). \tag{2.8}$$

The processes $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $\gamma = \gamma(t,z)$, $t \ge 0$, $z \in \mathbb{R}_0$, are predictable with respect to the right continuous filtration $\mathbb{F}$ on $(\Omega, \mathcal{F}, P)$. Here $\mathbb{F} := \{\mathcal{F}_t, t \ge 0\}$, is generated by B and N and $\mathcal{F}_0$ is augmented of all P-null events. Moreover it is assumed that

$$E\int_0^t \Big\{|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\nu(dz)\Big\}ds < \infty \quad P-a.s. \qquad \text{for all } t \ge 0. \tag{2.9}$$

In particular, solutions of stochastic differential equations of the form

$$dX(t) = \alpha(X(t))dt + \beta(X(t))dB(t) + \int_{\mathbb{R}} \gamma(X(t^-), z)\widetilde{N}(dt, dz) \tag{2.10}$$

are called *jump diffusions*. We refer to [A] and [S] for more information about Lévy processes and [ØS2] for the stochastic control of jump diffusions.

Since B and $\widetilde{N}$ are independent we may assume that

$$\Omega := \Omega_1 \times \Omega_2, \quad \mathcal{F} := \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}, \quad P := P_1 \times P_2 \tag{2.11}$$

where $(\Omega_i, \mathcal{F}^{(i)}, P_i)$, $i = 1, 2$, are the probability spaces generated by B and $\widetilde{N}$, respectively. Correspondingly we may consider $\mathbb{F}^{(i)} := \{\mathcal{F}_t^{(i)}, t \geq 0\}$, $i = 1, 2$, as the flow of events generated respectively be B and $\widetilde{N}$ and augmented of the events of null probability $P^{(i)}$, $i = 1, 2$, correspondingly. Then we can set $\mathcal{F}_t := \mathcal{F}_t^{(1)} \times \mathcal{F}_t^{(2)}$, $t \geq 0$.

From now on we fix a terminal time $T > 0$ constant.

In this context we will use two types of *Malliavin derivatives* of an $\mathcal{F}_T$-measurable random variable $F = F(\omega) = F(\omega_1, \omega_2)$, $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$:

(i) The Malliavin derivative $D_t F = \frac{\partial F}{\partial \omega_1}(t)$, with respect to B

(ii) The Malliavin derivative $D_{t,z} F = \frac{\partial F}{\partial \omega_2}(t, z)$, with respect to $\widetilde{N}$.

Both derivatives are defined in terms of the respective chaos expansions (see [I]) of F:

(i) For any $\omega_2 \in \Omega_2$ fixed, if $F = \sum_{n=0}^{\infty} I_n(f_n)$ is the chaos expansion of $F \in L_2(P)$ in terms of iterated integrals $I_n(f_n)$ of f_n, f_n being deterministic symmetric functions in $L_2(\lambda^n)$, with respect to B and $F \in \mathbb{D}_{1,2}^B$, i.e.

$$\|F\|_{\mathbb{D}_{1,2}^B}^2 := \sum_{n=1}^{\infty} nn! \|f_n\|_{L_2(\lambda^n)}^2 < \infty, \tag{2.12}$$

then we define

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \qquad t \in [0, T]. \tag{2.13}$$

Here λ denotes the Lebesgue measure on $[0, T]$. It is easily seen that if $F \in \mathbb{D}_{1,2}^B$, then

$$\|D_t F\|_{L^2(\lambda \times P)}^2 = \sum_{n=1}^{\infty} nn! \|f_n\|_{L_2(\lambda^n)}^2 = \|F\|_{\mathbb{D}_{1,2}^B}^2. \tag{2.14}$$

(ii) For any $\omega_1 \in \Omega_1$ fixed, if $F = \sum_{n=0}^{\infty} I_n(f_n)$ is the chaos expansion of $F \in L_2(P)$ in terms of iterated integrals $I_n(f_n)$ of f_n, f_n being deterministic

symmetric functions in
$L_2((\lambda \times \nu)^n)$, with respect to $\widetilde{N}$ and $F \in \mathbb{D}_{1,2}^{\widetilde{N}}$, i.e.

$$\|F\|^2_{\mathbb{D}_{1,2}^{\widetilde{N}}} := \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L_2((\lambda\times\nu)^n)} < \infty, \tag{2.15}$$

then we define

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, z)), \qquad (t, z) \in [0, T] \times \mathbb{R}_0. \tag{2.16}$$

In this case we get that if $F \in \mathbb{D}_{1,2}^{\widetilde{N}}$, then

$$\|D_{t,z}F\|^2_{L^2(\lambda\times\nu\times P)} = \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L_2((\lambda\times\nu)^n)} = \|F\|^2_{\mathbb{D}_{1,2}^{\widetilde{N}}}. \tag{2.17}$$

In this paper we will use the following two basic properties of the Malliavin derivatives:

Theorem 2.1. Duality formulae.

(i) (Cf. [N]). *Let $F \in \mathbb{D}^B_{1,2}$ and let $\varphi(t)$, $t \in [0,T]$, be an $\mathbb{F}$-predictable process satisfying*

$$E\Big[\int_0^T \varphi^2(t)dt\Big] < \infty.$$

Then

$$E\Big[F \cdot \int_0^T \varphi(t)dB(t)\Big] = E\Big[\int_0^T \varphi(t) \cdot D_t F \, dt\Big]. \tag{2.18}$$

(ii) (Cf. [BL], [DMØP]). *Let $F \in \mathbb{D}^{\widetilde{N}}_{1,2}$ and let $\psi(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$, be an $\mathbb{F}$-predictable process satisfying*

$$E\Big[\int_0^T \int_{\mathbb{R}_0} \psi^2(t,z)\nu(dz)dt\Big] < \infty.$$

Then

$$E\Big[F \cdot \int_0^T \int_{\mathbb{R}_0} \psi(t,z)d\widetilde{N}(t,z)\Big] = E\Big[\int_0^T \int_{\mathbb{R}_0} \psi(t,z) \cdot D_{t,z}F \, \nu(dz)dt\Big]. \tag{2.19}$$

We refer to e.g. [M], [N] for more information about Malliavin calculus in the Brownian motion case, and to e.g. [BL], [DMØP] and the forthcoming monograph [DØP] for the general Lévy process case. Different approaches to Malliavin calculus in the case of $\widetilde{N}$ are given in e.g. [NS] and [LSUV].

3. An optimal portfolio problem under partial information

We consider a market model on the complete probability space $(\Omega, \mathcal{F}, P)$ (see (2.11)) with finite time horizon $T > 0$ where two investments possibilities are available:

- a risk free asset with price dynamics

$$\begin{cases} dS_0(t) = \rho(t)S_0(t)dt, & t \in (0,T], \\ S_0(0) = 1 \end{cases} \tag{3.1}$$

- a risky asset with price dynamics

$$\begin{cases} dS_1(t) = S_1(t^-)\big[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0}\theta(t,z)\widetilde{N}(dt,dz)\big], & t \in (0,T], \\ S_1(0) > 0. \end{cases} \tag{3.2}$$

The driving noises in the model are the standard Brownian motion $B(t)$, $t \in [0,T]$, and the compensated Poisson random measure $\widetilde{N}(dt,dz)$, $(t,z) \in [0,T] \times \mathbb{R}_0$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Recall that $E\big[\widetilde{N}(dt,dz)^2\big] = \nu(dz)dt$, where $\nu(dz)$, $z \in \mathbb{R}_0$, is a σ-finite Borel measure which we assume to satisfy

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$$

- cf. (2.3)-(2.5). The parameters $\rho(t)$, $\mu(t)$, $\sigma(t)$ and $\theta(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$, are measurable, càglàd (i.e. left continuous paths with right sided limits) stochastic processes, adapted to the filtration $\mathbb{F} := \{\mathcal{F}_t \subseteq \mathcal{F}, t \in [0,T]\}$, see (2.11). We also assume that $\theta(t,z) > -1$, $\nu(dz)dt$-a.e. and that

$$E \int_0^T \Big\{ |\rho(t)| + |\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}_0} \theta^2(t,z)\nu(dz) \Big\} dt < \infty.$$

In this section we consider a dealer who has access only to some *partial information* of the market events. At any time t the σ-algebra $\mathcal{E}_t \subseteq \mathcal{F}_t$ represents the information at his disposal. We consider the collection of σ-algebras $\mathbb{E} := \{\mathcal{E}_t, t \in [0,T]\}$ to be a filtration.

We would like to note that dealers who are subjected to delays in receiving the information can also be regarded as having partial information. For example, in case $\delta > 0$ is a fixed *time delay*, at any time $t \geq \delta$ the actual information available would be $\mathcal{E}_t = \mathcal{F}_{t-\delta}$, while it would be $\mathcal{E}_t = \mathcal{E}_0$ for $t < \delta$.

In view of this lack of information, the portfolio $\pi(t)$ resulted from the dealer's decisions taken at time t is an $\mathcal{E}_t$-measurable random variable.

Taking the point of view of such a dealer we are interested in studying the optimization problem

$$u(x) := \sup_{\pi \in \mathcal{A}} E\left[U(X_\pi(T)) | X_\pi(0) = x\right] = E\left[U(X_{\pi^*}(T)) | X_\pi(0) = x\right], \tag{3.3}$$

for a given *utility function*

$$U : [0,\infty) \longrightarrow [-\infty, \infty)$$

that is an increasing, concave and lower semi-continuous function and which we assume to be continuously differentiable on $(0,\infty)$. The process $X_\pi(t)$, $t \in [0,T]$, given by

$$\begin{aligned} dX_\pi(t) = X_\pi(t^-)\Big\{ &\big[\rho(t) + \big(\mu(t) - \rho(t)\big)\pi(t)\big]dt \\ &+ \pi(t)\sigma(t)dB(t) + \int_{\mathbb{R}_0} \pi(t)\theta(t,z)\widetilde{N}(dt,dz)\Big\}, \end{aligned} \tag{3.4}$$

represents the *value process* of the portfolio π. The initial capital $X_\pi(0) = x > 0$ is fixed. Moreover we put

$$F_\pi(T) = U'(X_\pi(T))X_\pi(T), \qquad \text{for } \pi \in \mathcal{A}, \tag{3.5}$$

where $U'(u) = \frac{d}{du}U(u)$, $u > 0$.

We asssume that for all $t \in [0,T]$ there exists a family $\mathbb{A}_t$ of bounded $\mathcal{E}_t$-measurable and Malliavin differentiable random variables α such that the σ-algebra generated by $\mathbb{A}_t$ coincides with $\mathcal{E}_t$. Note that this is satisfied for example in the case of delayed information, i.e. $\mathcal{E}_t = \mathcal{F}_{t-\delta}$, for $t \geq \delta$, and $\mathcal{E}_t = \mathcal{E}_0$ for $t < \delta$.

The set $\mathcal{A}$ of *admissible portfolios* in (3.3) is characterized as follows:

(i) All the portfolios $\pi \in \mathcal{A}$ are measurable càglàd stochastic processes adapted to the filtration $\mathbb{E}$

(ii) $E\big[\int_0^T \big\{|\mu(s) - \rho(s)||\pi(s)| + \sigma^2(s)\pi^2(s) + \int_{\mathbb{R}_0}\pi^2(s)\theta^2(s,z)\nu(dz)\big\}ds\big] < \infty$, for all $\pi \in \mathcal{A}$,

(iii) for all $\pi \in \mathcal{A}$, $\pi(t)\theta(t,z) > -1 + \varepsilon_\pi$, $\nu(dz)dt$ - a.e. where $\varepsilon_\pi > 0$ may depend on π

(iv) for all $\pi \in \mathcal{A}$, $F_\pi(T)$ is Malliavin differentiable both with respect to the Brownian motion and with respect to the compensated Poisson random measure

(v) all the buy-hold-sell portfolios β, i.e.

$$\beta(s,\omega) := \alpha(\omega)\chi_{(t,t+h]}(s), \qquad s \in [0,T], \quad \omega \in \Omega,$$

with t, h such that $0 \leq t < t + h \leq T$ and $\alpha \in \mathbb{A}_t$, belong to $\mathcal{A}$.

(vi) For all portfolios $\pi, \beta \in \mathcal{A}$ with β bounded, there exists $\delta > 0$ such that $\pi + y\beta \in \mathcal{A}$ for all $y \in (-\delta, \delta)$.

(vii) Moreover we assume that for all $\pi, \beta \in \mathcal{A}$, with β bounded, there exists a $\delta > 0$ such that the family

$$\Big\{U'(X_{\pi+y\beta}(T))X_{\pi+y\beta}(T)\big|Y_{\pi+y\beta}(T)\big|\Big\}_{y\in(-\delta,\delta)}$$

is uniformly integrable. Here the stochastic process $Y_\pi(t)$, $t \in [0,T]$, is defined as

$$\begin{aligned} Y_\pi(t) &:= \int_0^t \Big\{\mu(s) - \rho(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}_0}\frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\nu(dz)\Big\}ds \\ &+ \int_0^t \sigma(s)dB(s) + \int_0^t\int_{\mathbb{R}_0}\frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(ds,dz), \quad t \in [0,T]. \end{aligned} \tag{3.6}$$

Our forthcoming discussion aims at giving a characterization of the locally optimal portfolios in (3.3).

Let us suppose that π gives a *local maximum for the problem* (3.3), in the sense that

$$E\big[U(X_{\pi+y\beta}(T))\big] \leq E\big[U(X_\pi(T))\big]$$

for all bounded $\beta \in \mathcal{A}$ and all $y \in (-\delta, \delta)$ where $\delta > 0$ is such that (vi) and (vii) hold. Let $\beta \in \mathcal{A}$ bounded and $\delta > 0$ such that (vi), (vii) hold. Then we have that $\pi + y\beta \in \mathcal{A}$ for $y \in (-\delta, \delta)$. For convenience let us define

$$g(y) := E\big[U\big(X_{\pi+y\beta}(T)\big)\big], \quad y \in (-\delta, \delta),$$

where, by the Itô formula,

$$\begin{aligned} X_\pi(T) = x \exp\Big\{ & \int_0^T \Big[\rho(s) + \big(\mu(s) - \rho(s)\big)\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\Big]ds \\ & - \int_0^T \int_{\mathbb{R}_0} \Big[\pi(s)\theta(s,z) - \ln\big(1 + \pi(s)\theta(s,z)\big)\Big]\nu(dz)ds \\ & + \int_0^T \pi(s)\sigma(s)dB(s) + \int_0^T \int_{\mathbb{R}_0} \ln\big(1 + \pi(s)\theta(s,z)\big)\widetilde{N}(ds,dz)\Big\} \end{aligned}$$

is the solution of (3.4). Since the function g is locally maximal at $y = 0$, thanks to (vi), we have that

$$\begin{aligned} 0 = & \frac{d}{dy} g(y)_{|y=0} \\ = & \; E\Big[U'(X_\pi(T))X_\pi(T)\Big\{ \int_0^T \beta(s)\big[\mu(s) - \rho(s) - \sigma^2(s)\pi(s) \\ & - \int_{\mathbb{R}_0} \big\{\theta(s,z) - \frac{\theta(s,z)}{1 + \pi(s)\theta(s,z)}\big\}\nu(dz)\big]ds \\ & + \int_0^T \beta(s)\sigma(s)dB(s) + \int_0^T \int_{\mathbb{R}_0} \frac{\beta(s)\theta(s,z)}{1 + \pi(s)\theta(s,z)}\widetilde{N}(ds,dz)\Big\}\Big]. \end{aligned}$$

Let us choose the portfolio $\beta \in \mathcal{A}$ to be of the form

$$\beta(s) = \alpha 1_{(t,t+h]}(s), \quad 0 \leq s \leq T,$$

with α as an arbitrary random variable in $\mathbb{A}_t$, for $t \in [0, T)$ and $h > 0$ such that $t + h \leq T$. Then from the equations above we have

$$\begin{aligned} 0 = E\Big[U'(X_\pi(T))X_\pi(T) \cdot \alpha \cdot \Big\{ & \int_t^{t+h} \big[\mu(s) - \rho(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\theta^2(s,z)}{1 + \pi(s)\theta(s,z)}\nu(dz)\big]ds \\ & + \int_t^{t+h} \sigma(s)dB(s) + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\theta(s,z)}{1 + \pi(s)\theta(s,z)}\widetilde{N}(ds,dz)\Big\}\Big] \end{aligned}$$

which holds for all $\alpha \in \mathbb{A}_t$. Thus we obtain that

$$E\big[F_\pi(T)\big(Y_\pi(t+h) - Y_\pi(t)\big)\cdot \alpha\big] = 0 \tag{3.7}$$

for the process $Y_\pi(t)$, $t \in [0,T]$, defined in (3.6) and $F_\pi(T)$ given in (3.5). By the duality formula, see Theorem 2.1, we have:

$$E\Big[F_\pi(T)\int_t^{t+h} \alpha\sigma(s)dB(s)\Big] = E\Big[\int_t^{t+h} \alpha\sigma(s)D_s\big(F_\pi(T)\big)ds\Big]. \tag{3.8}$$

and

$$E\Big[F_\pi(T)\int_t^{t+h}\int_{\mathbb{R}_0} \frac{\alpha\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(ds,dz)\Big] = E\Big[\int_t^{t+h}\int_{\mathbb{R}_0} \frac{\alpha\theta(s,z)D_{s,z}\big(F_\pi(T)\big)}{1+\pi(s)\theta(s,z)}\nu(dz)ds\Big]. \tag{3.9}$$

Substituting these two equations into (3.7) and noting that (3.7) holds for all $\alpha \in \mathbb{A}_t$, we obtain

$$E\Big[\int_t^{t+h}\Big\{\big(\mu(s)-\rho(s)-\sigma^2(s)\pi(s)\big)F_\pi(T) - \int_{\mathbb{R}_0}\frac{\pi(s)\theta^2(s,z)F_\pi(T)}{1+\pi(s)\theta(s,z)}\nu(dz) + \sigma(s)D_s\big(F_\pi(T)\big) + \int_{\mathbb{R}_0}\frac{\theta(s,z)D_{s,z}\big(F_\pi(T)\big)}{1+\pi(s)\theta(s,z)}\nu(dz)\Big\}ds\Big|\mathcal{E}_t\Big] = 0.$$

Since this holds for all t, h such that $0 \le t < t+h \le T$, we conclude that

$$E\Big[\big(\mu(s)-\rho(s)-\sigma^2(s)\pi(s)\big)F_\pi(T) + \sigma(s)D_s\big(F_\pi(T)\big)\Big|\mathcal{E}_s\Big] + E\Big[\int_{\mathbb{R}_0}\frac{\theta(s,z)D_{s,z}\big(F_\pi(T)\big) - \pi(s)\theta^2(s,z)F_\pi(T)}{1+\pi(s)\theta(s,z)}\nu(dz)\Big|\mathcal{E}_s\Big] = 0, \quad s \in [0,T]. \tag{3.10}$$

On the other side the argument can be reversed as follows. If we assume that (3.10) holds for some $\pi \in \mathcal{A}$, then

$$E\big[F_\pi(T)\big(Y_\pi(t+h) - Y_\pi(t)\big)|\mathcal{E}_t\big] = 0$$

which is equivalent to

$$E\big[F_\pi(T)\cdot\int_0^T \beta(s)\big[\mu(s)-\rho(s)-\sigma^2(s)\pi(s) - \int_{\mathbb{R}_0}\big\{\frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\big\}\nu(dz)\big]ds + \int_0^T\beta(s)\sigma(s)dB(s) + \int_0^T\int_{\mathbb{R}_0}\frac{\beta(s)\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(ds,dz)\big] = 0$$

for $\beta(s) = \alpha 1_{(t,t+h]}(s)$, $0 \le s \le T$, with $\alpha \in \mathbb{A}_t$. Thus the function $g(y) = E[U(X_{\pi+y\beta}(T)]$, $y \in (-\delta,\delta)$, satisfies

$$g'(0) = \frac{d}{dy}g(y)_{|y=0} = E\Big[U'(X_\pi(T)\frac{d}{dy}X_{\pi+y\beta}(T)_{|y=0}\Big] = 0$$

for all $\beta \in \mathcal{A}$. This is enough to conclude that $y = 0$ gives a local maximum for g and hence that $\pi \in \mathcal{A}$ is locally optimal for the problem (3.3). In fact, fix

$$\begin{aligned} h(y) := &\int_0^T \big\{\rho(s) + (\mu(s) - \rho(s))(\pi(s) + y\beta(s)) - \frac{1}{2}\sigma^2(s)(\pi(s) + y\beta(s))^2\big\}ds \\ &- \int_0^T \int_{\mathbb{R}_0} \big\{(\pi(s) + y\beta(s))\theta(s,z) - \log(1 + (\pi(s) + y\beta)\theta(s,z))\big\}\nu(dz)ds \\ &+ \int_0^T (\pi(s) + y\beta(s))\sigma(s)dB(s) + \int_0^T \int_{\mathbb{R}_0} \log(1 + (\pi(s) + y\beta)\theta(s,z))\widetilde{N}(ds,dz). \end{aligned}$$

Since U is increasing, the equation $g'(0) = 0$ gives that

$$\frac{d}{dy} X_{\pi+y\beta}(T)_{|y=0} = e^{h(0)}h'(0) = 0$$

which in particular leads to $h'(0) = 0$ and also

$$\frac{d^2}{dy^2} X_{\pi+y\beta}(T)_{|y=0} = e^{h(0)}h''(0) \leq 0,$$

being

$$h''(0) = -\int_0^T \sigma^2(s)\beta^2(s)ds - \int_0^T \int_{\mathbb{R}_0} \frac{\beta^2(s)\theta^2(s,z)}{(1 + \pi(s)\theta(s,z))^2} N(ds,dz) \leq 0.$$

Then we can conclude that

$$g''(0) = \frac{d^2}{dy^2} g(y)_{|y=0} = E\Big[U'(X_\pi(T) \cdot \frac{d^2}{dy^2} X_{\pi+y\beta}(T)_{|y=0}\Big] \leq 0$$

for all $\beta \in \mathcal{A}$.

Hence we have proved the following result.

Theorem 3.1. *A portfolio $\pi \in \mathcal{A}$ is a local maximum point for the problem* (3.3) *if and only if it satisfies the equation* (3.10).

Example 3.2. Let us consider the case with *logarithmic utility*, i.e.

$$U(u) = \log u, \qquad u > 0. \tag{3.11}$$

In this case we see that

$$F_\pi(T) = U'(X_\pi(T))X_\pi(T) = 1$$

and (3.10) simplifies to

$$\bar{\mu}(s) - \bar{\rho}(s) - \bar{\sigma}^2(s)\pi(s) - \pi(s)E\Big[\int_{\mathbb{R}_0} \frac{\theta^2(s,z)}{1 + \pi(s)\theta(s,z)}\nu(dz)\Big|\mathcal{E}_s\Big] = 0, \quad s \in [0,T], \tag{3.12}$$

where, for convenience in notation we have put $\bar{\mu}(s) := E[\mu(s)|\mathcal{E}_s]$, $\bar{\rho}(s) := E[\rho(s)|\mathcal{E}_s]$, $\bar{\sigma}^2(s) := E[\sigma^2(s)|\mathcal{E}_s]$, $s \in [0,T]$. This result is a synthesis of Corollary 2.13 and Theorem 3.12 in [ØS], where the logarithmic utility case is studied in the framework of an anticipative environment.

(i) In particular, if $\theta \equiv 0$ and $\sigma > 0$, then the process

$$\pi^*(s) = \frac{\bar{\mu}(s) - \bar{\rho}(s)}{\bar{\sigma^2}(s)}, \qquad s \in [0,T],$$

if belonging to $\mathcal{A}$, is an optimal portfolio for the problem (3.3) with logarithmic utility.

(ii) On the other hand, assume we have a price dynamics (3.2) driven by a Brownian motion and a centered Poisson process, i.e. $\sigma > 0$, $\nu(dz) = \delta_1(dz)$ and $\theta(t,z) = z$. In the case of logarithmic utility, the equation (3.10) gives

$$\bar{\mu} - \bar{\rho} - \bar{\sigma^2}\pi - \frac{\pi}{1+\pi} \equiv 0.$$

Thus the processes

$$\pi^* \equiv \frac{1}{2\bar{\sigma^2}^2}\Big\{\big(\bar{\mu} - \bar{\rho}\big) - \bar{\sigma^2} - 1 \pm \sqrt{\big[\big(\bar{\mu} - \bar{\rho}\big) + \bar{\sigma^2} - 1\big]^2 + 4\bar{\sigma^2}}\Big\},$$

if belonging to $\mathcal{A}$, are optimal for the problem (3.3).
In particular, in case the price dynamics of the risky asset was driven by the centered Poisson process only, i.e. $\sigma \equiv 0$ in this example, then (3.10) would lead to

$$\bar{\mu} - \bar{\rho}\pi - \frac{\pi}{1+\pi} \equiv 0.$$

Hence if

$$\pi^* \equiv \frac{\bar{\mu} - \bar{\rho}}{1 - \big(\bar{\mu} - \bar{\rho}\big)} \in \mathcal{A} \qquad (\bar{\mu} - \bar{\rho} \neq 1),$$

then it is optimal for (3.3).

Example 3.3. Next let us consider the case with *power utility*, i.e.

$$U(u) = \frac{1}{\gamma}u^\gamma, \quad u > 0, \tag{3.13}$$

where $\gamma \in (-\infty, 1) \setminus \{0\}$ is a constant. In this case we get $F_\pi(T) = X_\pi^\gamma(T)$, thus

$$D_t F_\pi(T) = \gamma X_\pi^\gamma(T) D_t \log X_\pi(T)$$

and

$$\begin{aligned} D_{t,z}F_\pi(T) &= \big(X_\pi(T) + D_{t,z}X_\pi(T)\big)^\gamma - X_\pi^\gamma(T) \\ &= X_\pi^\gamma(T)\big[\exp\big\{\gamma D_{t,z}\log X_\pi(T)\big\} - 1\big]. \end{aligned}$$

Therefore (3.10) becomes

$$
\begin{aligned}
&E\Big[X_\pi^\gamma(T)\Big(\mu(t)-\rho(t)-\sigma^2(t)\pi(t)\\
&+\int_{\mathbb{R}_0}\frac{\theta(t,z)}{1+\pi(t)\theta(t,z)}\big(\exp\big\{\gamma D_{t,z}\log X_\pi(T)\big\}-\pi(t)\theta(t,z)-1\big)\nu(dz)\\
&+\gamma\sigma(t)D_t\log X_\pi(T)\Big)\Big|\mathcal{E}_t\Big]=0.
\end{aligned}
\tag{3.14}
$$

In particular, if the coefficients $\mu(t),\rho(t),\sigma(t)$ and $\theta(t,z)$, $t\in[0,T]$, $z\in\mathbb{R}_0$, are all *deterministic* and we would like to have a $\pi(t)$, $t\in[0,T]$, deterministic also, then π must satisfy the equation

$$
\mu(t)-\rho(t)+(\gamma-1)\sigma^2(t)\pi(t)+\int_{\mathbb{R}_0}\theta(t,z)\big[(1+\pi(t)\theta(t,z))^{\gamma-1}-1\big]\nu(dz)=0.
\tag{3.15}
$$

Remark 3.4. The main feature of equation (3.10) is that it gives an explicit relation between the optimal portfolio π and the corresponding optimal terminal wealth $\hat{X}_\pi(T)=X_\pi(T)$. The following examples illustrate this.

Example 3.5. In the price dynamics, let us assume that $\theta=0$, $\sigma(t)\neq 0$, $t\in[0,T]$ and

$$
E\Big[\exp\Big\{\frac{1}{2}\int_0^T\big(\frac{\mu(s)-\rho(s)}{\sigma(s)}\big)^2ds\Big\}\Big]<\infty.
$$

Moreover we set $\mathcal{E}_t=\mathcal{F}_t$, for all t. In this context the market model is complete. It is known that the optimal terminal wealth $\hat{X}_\pi(T)$ is given by

$$
\hat{X}_\pi(T)=I(\mathcal{Y}(x)H_0(T)),
\tag{3.16}
$$

where $I:=(U')^{-1}$ is the inverse of $U'(u)=\frac{d}{du}U(u)$, and $\mathcal{Y}(x)=\mathcal{X}^{-1}(x)$ is the inverse of the function $\mathcal{X}$ defined by

$$
\mathcal{X}(y)=E\big[H_0(T)I(yH_0(T))\big],
\tag{3.17}
$$

with

$$
H_0(T)=\exp\Big\{-\int_0^T\frac{\mu(s)-\rho(s)}{\sigma(s)}dB(s)-\int_0^T\big[\rho(s)+\frac{1}{2}\big(\frac{\mu(s)-\rho(s)}{\sigma(s)}\big)^2\big]ds\Big\}
\tag{3.18}
$$

- see e.g. [KS], Chapter 3. Hence

$$
F_\pi(T)=U'(X_\pi(T))X_\pi(T)=\mathcal{Y}(x)H_0(T)I(\mathcal{Y}(x)H_0(T))
$$

(cf. (3.5)) is known in this case and, if $H_0(T) \in \mathbb{D}^B_{1,2}$, we can solve (3.10) for π as follows:

$$\pi(s) = \frac{\mu(s) - \rho(s)}{\sigma^2(s)} + \frac{E\big[D_s F_\pi(T)|\mathcal{F}_s\big]}{\sigma(s) E\big[F_\pi(T)|\mathcal{F}_s\big]}, \qquad s \in [0,T]. \tag{3.19}$$

Thus, if $\pi \in \mathcal{A}$, then it is optimal for the problem (3.3).

Example 3.6. Here we consider an extension of Example 3.5 to the general case (3.1)-(3.2) where $\sigma(t) \neq 0$, $\theta(t,z) \neq 0$, $t \in [0,T]$, $z \in \mathbb{R}_0$, and hence the market is possibly incomplete. As in the previous case we assume $\mathcal{E}_t = \mathcal{F}_t$, for all t.

Let $u(x)$ be as in (3.3) and consider the associated *dual problem*

$$v(y) := \inf_{Q \in \mathcal{M}_a} E\big[V\big(y\frac{dQ}{dP}\big)\big], \qquad y > 0, \tag{3.20}$$

where

$$V(\lambda) := \sup_{\xi \in \mathbb{R}} \big\{U(\xi) - \lambda\xi\big\}, \qquad \lambda > 0, \tag{3.21}$$

is the *Legendre transform* of U and $\mathcal{M}_a$ is the set of measures Q absolutely continuous with respect to P.

Then - under certain conditions - the optimal terminal wealth $\hat{X}_\pi(T)$ is given by

$$\hat{X}_\pi(T) = I\big(y(x)\frac{d\hat{Q}(y(x))}{dP}\big) \tag{3.22}$$

where $I := (U')^{-1}$ (as in Example 3.5). Here $x > 0$ is related to $y = y(x) > 0$ via $u'(x) = y$ or, equivalently, $x = -v'(y)$, and the measure $\hat{Q} = \hat{Q}(y) \in \mathcal{M}_a$ is the optimal measure for the dual problem (3.20). We refer to e.g. [KSch] and the survey [Sch] and the references therein for more details. Therefore, in terms of $\hat{Q}$ we get

$$F_\pi(T) = U'(\hat{X}_\pi(T))\hat{X}_\pi(T) = y(x)\frac{d\hat{Q}(y(x))}{dP} I\Big(y(x)\frac{d\hat{Q}(y(x))}{dP}\Big)$$

- cf. (3.5). With this expression for $F_\pi(T)$ in hands, we can see that an optimal portfolio $\pi \in \mathcal{A}$ must satisfy the following equation (see (3.10)):

$$\begin{aligned}&\big(\mu(s) - \rho(s) - \sigma^2(s)\pi(s)\big)E\big[F_\pi(T)\big|\mathcal{F}_s\big] + \sigma(s)E\big[D_s\big(F_\pi(T)\big)\big|\mathcal{F}_s\big] \\ &+ \int_{\mathbb{R}_0} \frac{\theta(s,z)E[D_{s,z}\big(F_\pi(T)\big)|\mathcal{F}_s] - \pi(s)\theta^2(s,z)E[F_\pi(T)|\mathcal{F}_s]}{1 + \pi(s)\theta(s,z)}\nu(dz) = 0, \quad s \in [0,T].\end{aligned} \tag{3.23}$$

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