

MAXIMAL ENTROPY SOLUTIONS FOR A SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX

SIDDHARTHA MISHRA

ABSTRACT. We study scalar conservation laws in one dimension with the flux function being discontinuous in the space variable. The existence and stability of infinitely many stable semi-groups of entropy solutions is shown. Each semi-group is characterized by an interface connection. The existence is a consequence of the convergence of a modified Engquist-Osher type scheme. A new concept of maximal entropy solutions is introduced in order to select physical relevant solutions. The maximal entropy solutions maximize the total entropy dissipated across the discontinuous interface and are shown to be unique.

1. INTRODUCTION

We are interested in the following one dimensional single conservation law,

$$(1.1) \quad \begin{aligned} u_t + (f(k(x), u))_x &= 0 \\ u(x, 0) &= u_0(x) \end{aligned}$$

With u as the unknown, f as the flux and k as a spatially dependent coefficient which may be discontinuous. The simplest form of (1.1) is the so-called 2-flux case given by,

$$(1.2) \quad u_t + (H(x)f(u) + (1 - H(x))g(u))_x = 0$$

Where H is the Heaviside function. For the rest of this paper, we will consider (1.2) for simplicity. All results obtained for it can be easily transferred to (1.1). See [4] for details.

Equations of the type (1.1) arise in a wide variety of applications like two-phase flows in a heterogeneous porous medium, the simulation of the Clarifier-Thickener unit used in waste water treatment plants and in traffic flow on highways with changing surface conditions. A detailed account of applications is provided in [10].

The main issues in the study of (1.1) is to formulate a suitable concept of entropy solutions for these equations and to investigate the existence and stability of entropy solutions. Another crucial issue is to derive suitable numerical schemes to approximate the entropy solutions. A lot of papers in recent years have addressed these questions in great detail, see [6],[5],[1],[7],[11],[12] and the references therein to name just a few.

One interesting feature of (1.1) was the discovery that it can give rise to different classes of meaningful entropy solutions in different physical regimes. In particular, the semi-group identified in [7] in the context of a vanishing viscosity limit was different from the one identified in [1] which corresponded to the physically meaningful solutions for two-phase flows in heterogeneous porous media. This surprising feature was formalized for (1.2) with convex fluxes in [2]. In [2], the authors were able to identify infinitely many L^1 stable semi-groups of entropy solutions for (1.1). Each of these semi-groups was characterized in terms of a interface connection.

Existence was proved by showing convergence for a Godunov type scheme and stability was based on a doubling of variables argument. The authors extended these results to general fluxes in [3] and to the more general equation (1.1) in [4].

Hence, the task of formulating a suitable concept of entropy solution for a given physical regime reduced to choosing a suitable connection. In [2], the authors proposed a concept of *optimal* entropy solutions for (1.2) based on optimizing a suitable functional at the interface. This functional was a measure of the total variation of the solution (measured in terms of a singular mapping) and the resulting optimizing connection turned out to be unique and coincided with the physically relevant non-undercompressive solution for two-phase flows in porous media. Yet, one undesirable feature of the interface functional was that it depended on the variation of the solution and didn't rely on any entropy criteria. This approach is also difficult to extend for more general flux geometries. See [3] for a discussion.

We propose a new approach to address the above problem in this paper. Our approach is based on defining a new functional depending on the entropy dissipated across the interface and we demand that not only is entropy dissipated across the interface, but we also choose the connection that dissipates the maximum entropy across the interface. For convex fluxes, this leads to a unique choice of the maximizer which corresponds to the *maximal* entropy solutions for (1.2). The maximal entropy solutions are easy to extend to more general fluxes and coincide with the non-undercompressive solution that is physically meaningful in the context of two-phase flows in a heterogeneous porous medium. In fact, we believe that a traveling wave analysis will lead to the same entropy criteria although the calculations are beyond the scope of this paper.

The situation above is similar to the one encountered while dealing with non-classical shocks resulting from vanishing diffusion-dissipation limits of conservation laws. See [8] for more details. In [8], there are infinitely many stable solutions and the physically relevant solution is selected by a maximization of the entropy dissipation measure. This motivates us to use a similar approach for (1.1).

Another novel feature of this paper is to derive a suitable modification of the Enquist-Osher scheme for approximating any semi-group of entropy solutions for (1.2). In [2], it was mentioned that we need Riemann solvers to be able to approximate these infinite classes of solutions but the modifications of the Enquist-Osher flux suggests that we can work with finite difference schemes in approximating these infinite semi-groups of solutions. The rest of this paper is organized as follows, In section (2), we state the hypothesis of the fluxes and identify the infinite classes of entropy solutions for (1.2). The Enquist-Osher flux is proposed and shown to converge in section (3) and the concept of maximal entropy solutions is described in section (4). We conclude with some numerical experiments in section (5).

2. *AB*-ENTROPY SOLUTIONS

Let $-\infty < s \leq S < \infty$. We consider the fluxes f, g with the following hypothesis,

$$H_1 \ . \ f, g \in \text{Lip}[s, S].$$

$$H_2 \ . \ f(s) = g(s), f(S) = g(S).$$

$$H_3 \ . \ g, f \text{ have at most one minimum (resp. maximum) and no maxima (resp. minima) in } [s, S].$$

The above hypothesis are standard for fluxes that arise in two-phase flows across changing rock types. See figure (1) for a typical example of the fluxes. We use

the standard definition of weak solutions of (1.2) and the standard Kruzhkov type interior entropy conditions. See section 2 of [2], definitions 2.1 and 2.3. We know that the interior entropy conditions are not enough to guarantee uniqueness. Hence as in [2], we proceed to define,

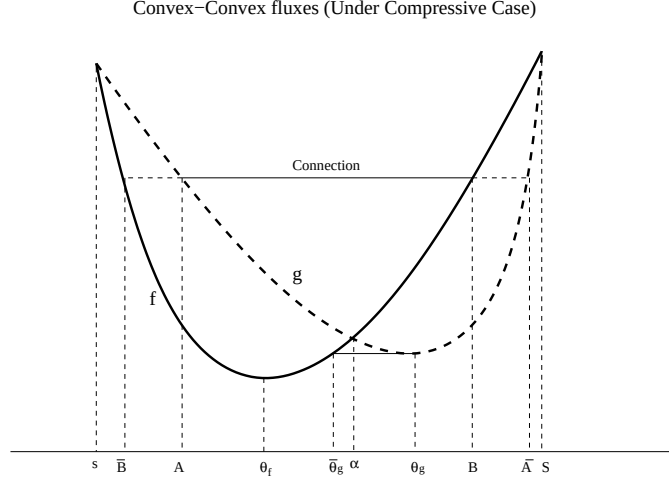


FIGURE 1. Shape of fluxes f, g and connections

Definition 2.1. *Connection* : In case the fluxes f and g satisfy the hypothesis (H_1, H_2, H_3) with θ_f be the unique minimum of f and θ_g being the unique minimum of g , then the pair $(A, B) \in [s, S]$ is said to be a connection if satisfies the following,

- (1). $g(A) = f(B)$.
- (2). $A \leq \theta_g$, and $\theta_f \geq B$.

It is said to be undercompressive if $A < \theta_g$ and $B > \theta_f$.

Some examples of connections are shown in figure (1)

Definition 2.2. *Interface entropy functional I_{AB}* : For any time t , we assume that the traces at the interface $u^-(t) = u(0-, t)$ and $u^+(t) = u(0+, t)$ exist. Then, we define the following Interface entropy functional relative to the connection (A, B) as

$$(2.1) \quad I_{AB}(t) = \text{sign}(u^-(t) - A)(g(u^-(t)) - g(A)) - \text{sign}(u^+(t) - B)(f(u^+(t)) - f(B))$$

Now, we are in position to define,

Definition 2.3. *Interface Entropy Condition*: We have the following interface entropy condition relative to a connection (A, B) ,

$$(2.2) \quad I_{AB}(t) \geq 0 \quad \text{a.e } t$$

Note that this condition is defined with respect to each connection (A, B) . Now for every choice of connection (A, B) , we are in position to define the AB -entropy solution as ,

Definition 2.4. *AB-Entropy solution*: A function $u \in L^\infty((-\infty, \infty) \times [0, \infty))$ is defined as the entropy solution of (1.2) relative to the connection (A, B) if the

following holds,

1. u is a weak solution of (1.2).
2. u satisfies the interior entropy conditions, and
3. u satisfies the interface entropy condition (2.2) relative to the connection (A, B) .

We state the following existence and stability theorem from [2](without proof).

Theorem 2.1. *Let the fluxes satisfy the hypothesis H_1, H_2 and H_3 and let the initial data $u_0 \in L^\infty(s, S) \cap BV(s, S)$, then for each choice of connection (A, B) , the corresponding AB-entropy solutions exist and are unique. Furthermore, let u, v be the AB-entropy solutions corresponding to data u_0, v_0 , then the following estimate holds,*

$$(2.3) \quad \int_{-\infty}^{\infty} |u(x, t) - v(x, t)| dx \leq \int_{-\infty}^{\infty} |u_0(x) - v_0(x)| dx$$

The proof of the above theorem is presented in detail in [2]. Existence is proved by showing convergence for Godunov-type schemes and stability follows by a Kruzhkov type doubling of variables argument. In the next section, we provide an alternative existence proof by showing convergence for a Enquist-Osher type scheme.

3. ENQUIST-OSHER TYPE SCHEME

We start with a few definitions,

Definition 3.1. *Let $h \in Lip[s, S]$ be a given function. Then, the standard Enquist-Osher flux corresponding to h is given by,*

$$(3.1) \quad H(a, b) = \frac{1}{2}(h(a) + h(b)) - \int_a^b |h'(\xi)| d\xi$$

For the special case where h has at most one minimum $\theta \in [s, S]$ and no maxima, (3.1) has a very simple form given by,

$$(3.2) \quad H(a, b) = h(\max(a, \theta)) + h(\min(\theta, b)) - h(\theta)$$

Next we need to define a flux at the interface below,

Definition 3.2. *Interface Enquist-Osher Flux: Let (A, B) be a connection for the fluxes f and g satisfying the hypothesis (H_1, H_2, H_3) , then the interface Enquist-Osher flux F_{AB} is defined as*

$$(3.3) \quad F_{AB}(a, b) = g(\max(a, A)) + f(\min(b, B)) - g(A)$$

Note that the above interface flux is a simple modification of the standard Enquist-Osher flux to this situation. If we take $(A, B) = (\theta_g, \overline{\theta_f})$, then when $f = g$, (3.3) reduces to the standard Enquist-Osher flux thus preserving the essential consistency.

We discretize in space on a uniform grid with mesh parameters $h > 0$ and define the space grid points x_i as follows.

$$x_i = \left(\frac{2i-1}{2} \right) h \quad \text{for } i \geq 1, \quad x_i = \left(\frac{2i+1}{2} \right) h \quad \text{for } i \leq -1.$$

Similarly we discretize uniformly in time with the time step Δt and denote the discrete time levels as t^n . The CFL condition is given by,

$$2 \frac{\Delta t}{h} M \leq 1$$

where $M = \max\{Lip(f), Lip(g)\}$. The Enquist-Osher scheme is then given by,

$$(3.4) \quad \begin{aligned} u_i^{n+1} &= u_i^n - \lambda(F(u_i^n, u_{i+1}^n) - F(u_{i-1}^n, u_i^n)) & \text{if } i \geq 2 \\ u_1^{n+1} &= u_1^n - \lambda(F(u_1^n, u_2^n) - F_{AB}(u_{-1}^n, u_1^n)) \\ u_{-1}^{n+1} &= u_{-1}^n - \lambda(F_{AB}(u_{-1}^n, u_1^n) - G(u_{-2}^n, u_{-1}^n)), \\ u_i^{n+1} &= u_i^n - \lambda(G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)) & \text{if } i \leq -2 \end{aligned}$$

Where F, G are the standard Enquist-Osher fluxes corresponding to f and g respectively and F_{AB} is the interface flux. We also need the approximating functions,

$$(3.5) \quad u_h(x, t) = u_i^n \text{ for } (x, t) \in [x_{i-1/2}, x_{i+1/2}) \times [n\Delta t, (n+1)\Delta t), \quad i \neq 0$$

We have the following convergence theorem,

Theorem 3.1. *For $h > 0$, let u_h be the corresponding calculated solution given by (3.4), Then there exists a subsequence $h_k \rightarrow 0$ such that u_{h_k} converges a.e. to a weak solution u of (1.2) satisfying interior entropy condition. Suppose the discontinuities of every limit function u of $\{u_h\}$ is a discrete set of Lipschitz curves; then $u_h \rightarrow u$ in $L_{loc}^\infty((0, \infty), L_{loc}^1(-\infty, \infty))$ as $h \rightarrow 0$, and u satisfies the interface entropy condition (2.2) relative to the connection (A, B) .*

Sketch of the Proof: The proof of convergence is based on a L^∞ estimate $s \leq u_i^n \leq S$ which follows from a invariant region principle. The key step is to use the singular mapping method to control the variation of the approximate solutions. This is presented in detail in the paper [9] for the Enquist-Osher scheme. An easy modification of the arguments in [9] apply in this case. The details of the calculation are not presented here.

4. MAXIMAL ENTROPY SOLUTIONS

Theorem (2.1) gives the existence of infinitely many stable semi-groups of solutions for (1.2). The problem of choosing one of these infinite classes is hard and depends on the physical regime being considered. As pointed out in the introduction, we will be presenting a new concept of solutions- the so called maximal entropy solutions. The key step is that in the proof of stability (hence uniqueness), we need a sign on the interface entropy functional I_{AB} (see (2.1)). This functional measures the entropy dissipated across the interface for each choice of connection and we want it to be non-negative for each admissible connection in order to obtain uniqueness. It is then reasonable to demand that instead of just asking for non-negativity, we want to maximize the maximum entropy dissipated across the interface for each connection among all admissible connections. This gives rise to the concept of a maximal connection defined below. First we need to define the admissible class of trace values,

Definition 4.1. *Admissible traces: The pair $u^-, u^+ \in [s, S]$ are defined as admissible traces for (1.2) with respect to the connection (A, B) if the following holds,*

1. $g(u^-) = f(u^+)$.
2. $I_{AB}(u^-, u^+) \geq 0$

The set of admissible traces with respect to the connection (A, B) is denoted by T_{AB}

Definition 4.2. *Maximal connection:* A connection (A_0, B_0) is defined to be a maximal entropy connection if the following holds,

$$(4.1) \quad \max_{u^-, u^+ \in T_{A_0 B_0}} I_{A_0 B_0}(u^-, u^+) = \max_{A, B} \max_{u^-, u^+ \in T_{AB}} I_{AB}(u^-, u^+)$$

The corresponding $A_0 B_0$ -entropy solution is defined as the maximal entropy solution of (1.2).

Therefore, for each choice of admissible connection we measure the maximum entropy dissipated at the interface and then maximize this quantity across all admissible connections. The characterization of the maximal entropy solutions is done in the following theorem,

Theorem 4.1. *Let f, g satisfy the hypothesis H_1, H_2 and H_3 , For simplicity, assume that $g(\theta_g) \geq f(\theta_f)$, then there exists a unique maximal connection given by,*

$$(A_0, B_0) = (\theta_g, \overline{\theta_f})$$

where $\overline{\theta_f} > \theta_f$ such that $g(\theta_g) = f(\overline{\theta_f})$

Note that the maximal entropy solution coincides with the physically relevant vanishing capillarity solution for two-phase flows in a heterogeneous porous medium.

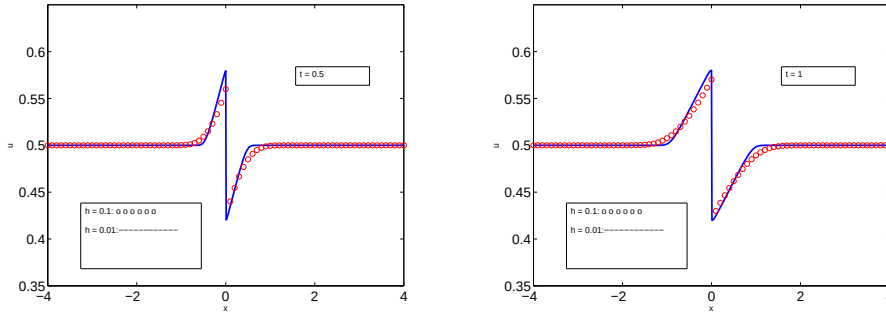


FIGURE 2. Approximate solutions by the Enquist-Osher scheme for the maximal entropy solutions with $h = 0.1$ and $h = 0.01$ at times $t = 0.5$ (left) and $t = 1$ (right)

Proof. First we characterize the admissible traces in the claim below.

Claim: If $u^-, u^+ \in T_{AB}$, then $g(u^-) \geq g(A)$.

In order to prove the above claim, we proceed by contradiction. Let $g(u^-) < g(A)$. Then as $(u^-, u^+) \in T_{AB}$, we have that

$$I_{AB}(u^-, u^+) = (\text{sign}(u^- - A) - \text{sign}(u^+ - B))(g(u^-) - g(A)) \geq 0$$

This implies that $u^- \leq A$ and $u^+ > B$. But as g is decreasing in (s, A) , this implies that $g(u^-) > g(A)$ which is a contradiction thus proving the claim.

It is easy to check that for each connection (A, B) and for the given flux geometry, we have

$$\max_{(u^-, u^+) \in T_{AB}} I_{AB}(u^-, u^+) = \max\{g(s), g(S)\} - g(A)$$

Now, when we maximize over all connections (A, B) , we optimize the above quantity when $A = \theta_g$. Hence $(\theta_g, \overline{\theta_f})$ is the maximal entropy connection in this case. $\square$ $\square$

5. NUMERICAL EXPERIMENTS

We present a simple numerical experiment with the following fluxes and data,

$$\begin{aligned} f(u) &= \frac{2u(u-1)}{u+1} \quad , \quad g(u) = \frac{2u(u-1)}{2-u} \\ u(x, 0) &\equiv 0.5 \end{aligned}$$

The above fluxes represent flows across two rock-types for two phase separated at $x = 0$. The initial mixture of phases is redistributed by gravity. We show the solutions obtained with an Enquist-Osher scheme (3.4) for the maximal entropy connection in figure (2). As expected, the schemes gives good resolution and converges to the maximal solution.

REFERENCES

- [1] Adimurthi, J. Jaffre and G. D. Veerappa Gowda. Godunov type methods for Scalar Conservation Laws with Flux function discontinuous in the space variable. *SIAM J. Numer. Anal.*, 42 (1): 179-208, 2004.
- [2] Adimurthi, Siddhartha Mishra and G.D. Veerappa Gowda. Optimal entropy solutions for conservation laws with discontinuous flux. *Journal of Hyp. Diff. Eqns.*, 2 (4): 787-838, 2005.
- [3] Adimurthi, Siddhartha Mishra and G.D. Veerappa Gowda. Conservation laws with flux function discontinuous in the space variable - III, The general case. *Preprint*, 2005.
- [4] Adimurthi, Siddhartha Mishra and G.D. Veerappa Gowda. Convergence of Godunov type methods for conservation laws with spatially varying flux functions. *To appear in Math. Comp*
- [5] R. Bürger, K.H. Karlsen, N.H. Risebro and J.D. Towers. Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier thickener units. *Numer. Math.*, 97 (1):25-65, 2004.
- [6] T. Gimse and N.H. Risebro. Solution of Cauchy problem for a conservation law with discontinuous flux function. *SIAM J. Math. Anal.*, 23 (3): 635-648, 1992.
- [7] K. H. Karlsen, N.H. Risebro and J.D. Towers. L^1 stability for entropy solution of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients. *Skr. K. Nor. Vidensk. Selsk.*, 3, 2003, 49 pages.
- [8] P. LeFloch. Hyperbolic systems of conservation laws: The theory of classical and non-classical shock waves. Birkhauser, Basel, 2002.
- [9] S. Mishra. Convergence of upwind finite difference schemes for a scalar conservation law with indefinite discontinuities in the flux function. *SIAM J. Num. Anal.*, 43(2): 559- 577, 2005.
- [10] S. Mishra. Analysis and Numerical approximation of conservation laws with discontinuous coefficients *PhD Thesis*, Indian Institute of Science, Bangalore, 2005.
- [11] J.D. Towers. Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM J. Numer. Anal.*, 38(2):681-698, 2000.
- [12] J.D. Towers. A difference scheme for conservation laws with a discontinuous flux-the nonconvex case. *SIAM J. Numer. Anal.*, 39(4): 1197-1218, 2001.

(Siddhartha Mishra)

CENTRE OF MATHEMATICS FOR APPLICATIONS (CMA)

UNIVERSITY OF OSLO

P.O. BOX 1053, BLINDERN

N-0316 OSLO, NORWAY

E-mail address: `siddharm@cma.uio.no`