Rigidity in motivic homotopy theory

Oliver Röndigs and Paul Arne Østvær

March 13, 2007

Abstract

We show that extensions of algebraically closed fields induce full and faithful functors between the respective motivic stable homotopy categories with finite coefficients.

Contents

1 Introduction **2**

2 Transfer maps **3**
2.1 Construction of transfer maps 3
2.2 Properties of transfer maps 5
2.3 An alternate approach 9

3 Moore spectra **11**

4 Motivic rigidity **14**

A Homological localization **21**
A.1 A fibrant replacement 21
A.2 The local model structure 23

1 Introduction

This paper is concerned with rigidity in motivic stable homotopy theory. Our main result compares mod-ℓ motivic stable homotopy categories under extensions of algebraically closed fields.

Theorem: *Suppose K/k is an extension of algebraically closed fields and ℓ is prime to the exponential characteristic of k. Then base change defines a full and faithful functor $\mathbf{SH}(k)^\ell \longrightarrow \mathbf{SH}(K)^\ell$ between mod-ℓ motivic stable homotopy categories.*

The proof we give of the motivic rigidity theorem uses transfer maps in motivic stable homotopy theory and a homological localization theory for motivic symmetric spectra. In Section 2 we construct such transfer maps for linearly trivial maps over general base schemes, and prove certain compatibility results with respect to Thom spaces of vector bundles. Next, in Section 3, we introduce mod-ℓ motivic stable homotopy categories by localizing with respect to mod-ℓ motivic Moore spectra. This construction relies on a widely applicable localization theory for motivic symmetric spectra, see Appendix A. Finally the algebraically closed field assumption enters in the construction of a map from the group of divisors $\mathrm{Div}(C)$ for a smooth affine curve C to $\mathrm{Hom}_{\mathbf{SH}(k)}(k_+, C_+)$. A combination of the algebro-geometric input in Suslin's proof of rigidity for algebraic K-groups [16] and subsequent generalizations, for example [14], and an explicit fibrant replacement functor in the underlying mod-ℓ model structure allows to finish the proof.

It turns out that the same approach leads to rigidity results for mod-ℓ reductions of certain motivic symmetric spectra. Motivic cohomology is a particularly interesting example of such a spectrum, in which case the theory specializes to a rigidity theorem for categories of motives. Rather than enmeshing the introduction with technical details, we refer to Section 4 for precise statements of these results.

The authors gratefully acknowledge the excellent working conditions and support provided by the Fields Institute during the spring 2007 Thematic Program on Geometric Applications of Homotopy Theory.

Conventions and notations. Recall the Tate object T is the smash product of the simplicial circle S^1 and the multiplicative group $(\mathbf{G}_m, 1)$ pointed by the unit section. It is the preferred suspension coordinate in the category of motivic symmetric spectra $\mathbf{MSS}_S$ relative to a noetherian base scheme S of finite Krull dimension.

We denote the pointed motivic unstable homotopy category of S by $\mathbf{H}(S)$, see [12], the motivic stable homotopy category of S by $\mathbf{SH}(S)$, see [10], blow-ups by $\mathbf{Bl}$, normal bundles by $\mathcal{N}$, projectivizations by $\mathbf{P}$, tangent bundles by $\mathcal{T}$, and the Thom space of a vector bundle $p\colon V \longrightarrow\!\!\!\!\!\rightarrow Y$ equipped with a zero section p_0 by $\mathbf{Th}(p) \equiv V/V \smallsetminus p_0(Y)$. The Tate object can be identified with the Thom space of the trivial line bundle $\mathbf{A}^1$. Internal hom objects in some closed symmetric monoidal category are denoted by $\underline{\mathrm{Hom}}$. Throughout we use the motivic model structure on categories of motivic spaces in [4]. Finally all the diagrams in this paper are commutative.

2 Transfer maps

In this section we construct transfer maps in the motivic stable homotopy category over a general base scheme, prove some basic properties required in the proof of the motivic rigidity theorem, and outline an alternate construction of transfers for finite étale maps.

2.1 Construction of transfer maps

Definition 2.1: A map $f\colon X \longrightarrow\!\!\!\!\!\rightarrow Y$ in the category Sm_S of smooth S-schemes of finite type is linear if it admits a factorization

$$X \overset{i}{\hookrightarrow\!\!\!\!\!\rightarrow} V \overset{p}{\longrightarrow\!\!\!\!\!\rightarrow} Y,$$

where i is a closed embedding, defined by some quasi-coherent sheaf of ideals $\mathcal{I} \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_V$, and p is a vector bundle. A map is linearly trivial if there exists a linearization (i, p) such that both $\mathcal{N}i \equiv \underline{\mathrm{Hom}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \longrightarrow\!\!\!\!\!\rightarrow X$ and p are isomorphic to trivial vector bundles. A linear trivialization consists of a linearization together with choices of trivializations $\theta\colon \mathcal{N}i \overset{\cong}{\longrightarrow\!\!\!\!\!\rightarrow} X \times \mathbf{A}^m$ and $\rho\colon Y \times \mathbf{A}^n \overset{\cong}{\longrightarrow\!\!\!\!\!\rightarrow} p$. See also [19].

Example 2.2: A map of finite type between finitely generated algebras is linear and every finite separable field extension is linearly trivial by the primitive element theorem.

The next result follows immediately from [6, B.7.4].

Proposition 2.3: *1. Linear maps are preserved under base change.*

2. Linearly trivial maps are preserved under base change along flat maps.

Fix a map $f\colon X \longrightarrow Y$ of relative dimension d with linear trivialization (i, p, θ, ρ) such that if p has rank n, then $\mathcal{N}i$ has rank $n-d$. If $W \equiv V \oplus \mathbf{A}^1 \longrightarrow Y$ is the direct sum of p and the trivial line bundle, there is an open embedding $j\colon V \hookrightarrow \mathbf{P}(W)$ with corresponding closed complement $\mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$. The composition of p_0 and j gives a Y-rational point 0 on $\mathbf{P}(W)$ and a diagram:

$$\begin{array}{ccccccccc} V \smallsetminus \{0\} & \hookrightarrow & \mathbf{P}(W) \smallsetminus \{0\} & \hookleftarrow & \mathbf{P}(V) & \hookrightarrow & \mathbf{P}(W) \smallsetminus j \circ i(X) & \hookleftarrow & V \smallsetminus i(X) \\ \downarrow & D1 & \downarrow & D2 & \downarrow & D3 & \downarrow & D4 & \downarrow \\ V & \hookrightarrow & \mathbf{P}(W) & = & \mathbf{P}(W) & = & \mathbf{P}(W) & \hookleftarrow & V \end{array} \tag{1}$$

Since $D1$ and $D4$ are Nisnevich distinguished squares [12, 3.1.3], the induced quotient maps $V/V \smallsetminus \{0\} \longrightarrow \mathbf{P}(W)/\mathbf{P}(W) \smallsetminus \{0\}$ and $V/V \smallsetminus i(X) \longrightarrow \mathbf{P}(W)/\mathbf{P}(W) \smallsetminus j \circ i(X)$ are weak equivalences. Moreover, since the closed embedding $\mathbf{P}(V) \hookrightarrow \mathbf{P}(W) \smallsetminus \{0\}$ is the zero section of the canonical quotient line bundle $\mathcal{O}_{\mathbf{P}(V)}(1)$ on $\mathbf{P}(V)$ it is a strict $\mathbf{A}^1$-homotopy equivalence [12, 3.2.2], so that the square $D2$ induces a weak equivalence of pointed quotient motivic spaces $\mathbf{P}(W)/\mathbf{P}(V) \longrightarrow \mathbf{P}(W)/\mathbf{P}(W) \smallsetminus \{0\}$. Using square $D3$ we conclude there exists a map $\mathbf{Th}(p) \longrightarrow V/V \smallsetminus i(X)$ in $\mathbf{H}(S)$, which combined with the homotopy purity isomorphism $V/V \smallsetminus i(X) \longrightarrow \mathbf{Th}(\mathcal{N}i)$ in [12, 3.2.23] induces

$$(i, p)^!\colon \mathbf{Th}(p) \longrightarrow \mathbf{Th}(\mathcal{N}i).$$

The maps θ and ρ induce isomorphisms $\mathbf{Th}(\mathcal{N}i) \xrightarrow{\cong} X_+ \wedge T^{n-d}$ and $Y_+ \wedge T^n \xrightarrow{\cong} \mathbf{Th}(p)$ of pointed motivic spaces by [12, 3.2.17]. Now using $(i, p)^!$ and taking suspension spectra we get a map $Y_+ \wedge T^n \longrightarrow X_+ \wedge T^{n-d}$ in $\mathbf{SH}(S)$. Since smashing with the Tate object is an isomorphism in the motivic stable homotopy category, there exists a map

$$(i, p, \theta, \rho)^!\colon Y_+ \wedge T^d \longrightarrow X_+.$$

The properties we require of these types of transfer maps are proved in the next section.

Remark 2.4: The map $(i, p)^!$ does not only depend on $p \circ i$ in general. For example, the identity map on the projective line factors through the zero sections i_0 and i_1 of the trivial vector bundle $\mathcal{O}_{\mathbf{P}^1}$ and the canonical invertible sheaf $\mathcal{O}_{\mathbf{P}^1}(1)$ respectively. Lemma 2.5 shows the corresponding maps between Thom spaces $(i_0, p_0)^!$ and $(i_1, p_1)^!$ are isomorphisms. However, the Thom spaces of $\mathcal{O}_{\mathbf{P}^1}$ and $\mathcal{O}_{\mathbf{P}^1}(1)$ have distinct motivic stable homotopy types since $\mathbf{Th}(\mathcal{O}_{\mathbf{P}^1}(1)) \cong \mathbf{P}^2_+$ and $\mathbf{Th}(\mathcal{O}_{\mathbf{P}^1}) = T \wedge \mathbf{P}^1_+$. The Steenrod square $\mathrm{Sq}^{2,1}$ acts non-trivially on $\mathbf{P}^2_+$, but trivially on the suspension $T \wedge \mathbf{P}^1_+$.

2.2 Properties of transfer maps

The caveat Remark 2.4 relies on the next lemma which is a slight variant of Voevodsky's [18, 2.2]. We sketch a proof for the sake of introducing notation.

Lemma 2.5: *If the closed embedding i is the zero section of p, then $(i,p)^!$ coincides with the map of Thom spaces induced by the natural isomorphism $p \cong \mathcal{N}i$.*

Proof. The assumption implies that $D2$ coincides with $D3$ and $D1$ coincides with $D4$. Hence (1) induces the identity map. The homotopy purity isomorphism [12, 3.2.23] for a smooth pair $i\colon X \hookrightarrow\!\!\!\!\rightarrow V$ over S involves the blow-up $\mathbf{Bl}(i)$ of $V \times \mathbf{A}^1$ along $i(X) \times \{0\}$. There is a canonical closed embedding $y\colon X \times \mathbf{A}^1 \hookrightarrow\!\!\!\!\rightarrow \mathbf{Bl}(i)$ and the normal bundle of $i(X) \times \{0\} \hookrightarrow\!\!\!\!\rightarrow V \times \mathbf{A}^1$ is isomorphic to $\mathcal{N}i \oplus \mathbf{A}^1_X$. Then the diagram

$$\begin{array}{ccccccc}
V \smallsetminus i(X) & \hookrightarrow\!\!\!\!\rightarrow & \mathbf{Bl}(i) \smallsetminus y(X \times \mathbf{A}^1) & \leftarrow\!\!\!\!\supset & \mathbf{P}(\mathcal{N}i \oplus \mathbf{A}^1_X) \smallsetminus \mathbf{P}(\mathbf{A}^1_X) & \multimap & \mathcal{N}i \smallsetminus z(X) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
V & \hookrightarrow\!\!\!\!\rightarrow & \mathbf{Bl}(i) & \leftarrow\!\!\!\!\supset & \mathbf{P}(\mathcal{N}i \oplus \mathbf{A}^1_X) & \multimap & \mathcal{N}i
\end{array}$$

where $z\colon X \hookrightarrow\!\!\!\!\rightarrow \mathcal{N}i$ denotes the zero section induces a zig-zag of weak equivalences

$$\begin{array}{rcl}
V/V \smallsetminus i(X) & \xrightarrow{\sim} & \mathbf{Bl}(i)/\mathbf{Bl}(i) \smallsetminus y(X \times \mathbf{A}^1) \\
& \xleftarrow{\sim} & \mathbf{P}(\mathcal{N}i \oplus \mathbf{A}^1_X)/\mathbf{P}(\mathcal{N}i \oplus \mathbf{A}^1_X) \smallsetminus \mathbf{P}(\mathbf{A}^1_X) \\
& \xleftarrow{\sim} & \mathcal{N}i/\mathcal{N}i \smallsetminus z(X).
\end{array}$$

Now if i is the zero section of p, then $\mathbf{Bl}(i)$ is the total space of the tautological line bundle $\mathcal{O}_{\mathbf{P}(V \oplus \mathbf{A}^1_X)}(-1)$ and there are canonical maps from the pointed motivic spaces in the zig-zag of weak equivalences to $\mathbf{P}(V \oplus \mathbf{A}^1_X)/\mathbf{P}(V \oplus \mathbf{A}^1_X) \smallsetminus \mathbf{P}(\mathbf{A}^1_X)$, which induce sheaf isomorphisms at $V/V \smallsetminus i(X)$ and $\mathcal{N}i/\mathcal{N}i \smallsetminus z(X)$ [12, 3.2.17]. And $p \cong \mathcal{N}i$ is the naturally induced isomorphism of Nisnevich sheaves. □

Lemma 2.5 shows that if the identity map id_X factors via the zero section i of some vector bundle p of rank n, then $(i,p,\theta,\rho)^!$ depends only on the linear trivialization (θ,ρ) in the sense that the isomorphism $X_+ \wedge T^n \longrightarrow X_+ \wedge T^n$ is induced by the automorphism $\theta \circ (p \cong \mathcal{N}i) \circ \rho$ of $\mathbf{A}^n_X$. Therefore, every linear trivialization of the identity map on X corresponds to the choice of an element in the image of the induced map

$$\phi(X)\colon \mathbf{GL}_n(X) \longrightarrow \mathrm{Aut}_{\mathbf{SH}(S)}(X_+). \tag{2}$$

Remark 2.6: If every $n \times n$ matrix in X with determinant 1 is a product of elementary matrices and $e_{ij}(a)$ is an elementary matrix, the linear homotopies $\big(e_{ij}(a), t\big) \longmapsto e_{ij}(at)$ imply that the composite $\mathbf{SL}_n(X) \hookrightarrow \mathbf{GL}_n(X) \xrightarrow{\phi(X)} \mathrm{Aut}_{\mathbf{SH}(S)}(X_+)$ is the trivial map.

Lemma 2.7: *Suppose (i,p) and (i',p') are linearizations and there exists a diagram in Sm_S consisting of pullback components:*

$$\begin{array}{ccccc} & & i' & & \\ X' & \hookrightarrow\!\!\!\!\rightarrow & V' & \xrightarrow{p'} & Y' \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ & {\scriptstyle i} & & {\scriptstyle p} & \\ X & \hookrightarrow\!\!\!\!\rightarrow & V & \xrightarrow{} & Y \end{array}$$

If the canonical map of total spaces $\gamma\colon \mathcal{N}i' \longrightarrow g^*\mathcal{N}i$ is an isomorphism of vector bundles over X', for example if h is flat [6, B.7.4], then there is a naturally induced diagram in $\mathbf{H}(S)$:

$$\begin{array}{ccc} \mathbf{Th}(p') & \xrightarrow{(i',p')^!} & \mathbf{Th}(\mathcal{N}i') \\ \downarrow & & \downarrow \\ \mathbf{Th}(p) & \xrightarrow{(i,p)^!} & \mathbf{Th}(\mathcal{N}i) \end{array}$$

Proof. It suffices to check commutativity for the two maps employed in the definition of $(i,p)^!$. For $\mathbf{Th}(p) \longrightarrow V/V \smallsetminus i(X)$ this follows using compatibility of the embeddings $V \hookrightarrow\!\!\!\circ\!\!\!\rightarrow W$ and $V' \hookrightarrow\!\!\!\circ\!\!\!\rightarrow W'$, while for $V/V \smallsetminus i(X) \longrightarrow \mathbf{Th}(\mathcal{N}i)$ one uses the setup in the proof of the homotopy purity isomorphism, cf. [18, 2.1] and Lemma 2.5. □

Corollary 2.8: *Assumptions being as in Lemma 2.7, then provided (θ,ρ) and (θ',ρ') are compatible trivializations the corresponding transfer maps induce a diagram in $\mathbf{SH}(S)$:*

$$\begin{array}{ccc} Y'_+ \wedge T^d & \xrightarrow{(i',p',\theta',\rho')^!} & X'_+ \\ {\scriptstyle h_+ \wedge T^d}\downarrow & & \downarrow{\scriptstyle g_+} \\ Y_+ \wedge T^d & \xrightarrow{(i,p,\theta,\rho)^!} & X_+ \end{array}$$

Proof. This follows from Lemma 2.7 and the assumption on the trivializations. □

Lemma 2.9: *Suppose $X = X^0 \coprod X^1$ is the disjoint union of connected schemes and (i,p) is a linearization of some map $f\colon X \longrightarrow Y$ in Sm_S. Let (i^0,p) and (i^1,p) be the induced linearizations of $f^0\colon X^0 \hookrightarrow X \longrightarrow Y$ and $f^1\colon X^1 \hookrightarrow X \longrightarrow Y$. Then there is a diagram in $\mathbf{SH}(S)$:*

$$\begin{array}{ccc} \mathbf{Th}(p) & \xrightarrow{\left((i^0,p)^!,(i^1,p)^!\right)} & \mathbf{Th}(\mathcal{N}i^0)\times\mathbf{Th}(\mathcal{N}i^1) \\ {\scriptstyle (i,p)^!}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbf{Th}(\mathcal{N}i) & \xleftarrow{\nabla} \mathbf{Th}(\mathcal{N}i)\vee\mathbf{Th}(\mathcal{N}i) & \xleftarrow{} \mathbf{Th}(\mathcal{N}i^0)\vee\mathbf{Th}(\mathcal{N}i^1) \end{array}$$

Proof. The map $\mathbf{Th}(\mathcal{N}i^n) \longrightarrow \mathbf{Th}(\mathcal{N}i)$ is induced by the inclusion of X^n into X, and ∇ is the codiagonal map. We begin with some remarks on pullbacks in Sm_S of the form:

$$\begin{array}{ccc} U\cap V & \hookrightarrow & V \\ \downarrow & \lrcorner & \downarrow \\ U & \hookrightarrow & Z \end{array} \tag{3}$$

The monomorphisms $V/U\cap V \hookrightarrow Z/U\cap V$ and $U/U\cap V \hookrightarrow Z/U\cap V$ induce a map $\phi\colon V/U\cap V\vee U/U\cap V \longrightarrow Z/U\cap V$. And since (3) is a pullback, ϕ is a monomorphism. If $Z = U\cup V$, then (3) is a homotopy pushout and hence ϕ is a weak equivalence. The maps $Z/U\cap V \longrightarrow Z/U, Z/V$ induce a map $\psi\colon Z/U\cap V \longrightarrow Z/U\times Z/V$ and the composite $V/U\cap V\vee U/U\cap V \hookrightarrow Z/U\cap V \longrightarrow Z/U\times Z/V$ coincides with the canonical map $V/U\cap V\vee U/U\cap V \longrightarrow Z/U\vee Z/V \longrightarrow Z/U\times Z/V$. If $Z = U\cup V$, the map between the wedge products is a weak equivalence. It follows that $\psi\phi$ is a weak equivalence and likewise for ψ by saturation of weak equivalences. Clearly these maps are natural with respect to natural transformations between squares of the form (3).

Following the notation in Lemma 2.5 we now consider the natural transformations:

$$\begin{array}{ccccc} V\smallsetminus 0 = V\smallsetminus 0 & & \mathbf{P}(W)\smallsetminus 0 = \mathbf{P}(W)\smallsetminus 0 & & \mathbf{P}(W)\smallsetminus\mathbf{P}(V) = \mathbf{P}(W)\smallsetminus\mathbf{P}(V) \\ \| \quad \downarrow & & \| \quad \downarrow & & \| \quad \downarrow \\ V\smallsetminus 0 \hookrightarrow V & & \mathbf{P}(W)\smallsetminus 0 \hookrightarrow \mathbf{P}(W) & & \mathbf{P}(W)\smallsetminus\mathbf{P}(V) \hookrightarrow \mathbf{P}(W) \end{array}$$

$$\begin{array}{ccc} V\smallsetminus i(X) \hookrightarrow V\smallsetminus i^0(X^0) & & \mathbf{P}(W)\smallsetminus j\circ i(X) \hookrightarrow \mathbf{P}(W)\smallsetminus j\circ i^0(X^0) \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ V\smallsetminus i^1(X^1) \hookrightarrow V & & \mathbf{P}(W)\smallsetminus j\circ i^1(X^1) \hookrightarrow \mathbf{P}(W) \end{array}$$

Note there is an induced diagram in $\mathbf{SH}(S)$:

$$\begin{array}{ccc} \mathbf{Th}(p) & \longrightarrow & V/V \smallsetminus i(X) \\ \downarrow & & \downarrow \cong \\ \mathbf{Th}(p) \times \mathbf{Th}(p) & \to & \big(V/V \smallsetminus i^0(X^0)\big) \times \big(V/V \smallsetminus i^1(X^1)\big) \end{array} \tag{4}$$

Here, the left hand vertical map is the diagonal and the isomorphism coincides with the composition of the canonical zig-zag of isomorphisms

$$V/V \smallsetminus i(X) \xleftarrow{\cong} \big(V/V \smallsetminus i^0(X^0)\big) \vee \big(V/V \smallsetminus i^1(X^1)\big) \xrightarrow{\cong} \big(V/V \smallsetminus i^0(X^0)\big) \times \big(V/V \smallsetminus i^1(X^1)\big).$$

Analogously, using the natural transformations

$$\begin{array}{ccc} V \smallsetminus i(X) & \hookrightarrow & V \smallsetminus i^0(X^0) \\ \downarrow & & \downarrow \\ V \smallsetminus i^1(X^1) & \hookrightarrow & V \end{array} \qquad \begin{array}{ccc} \mathbf{Bl}(i) \smallsetminus y(\mathbf{A}^1_X) & \hookrightarrow & \mathbf{Bl}(i) \smallsetminus y^0(\mathbf{A}^1_{X^0}) \\ \downarrow & & \downarrow \\ \mathbf{Bl}(i) \smallsetminus y^1(\mathbf{A}^1_{X^1}) & \hookrightarrow & \mathbf{Bl}(i) \end{array} \qquad \begin{array}{ccc} \mathcal{N}i \smallsetminus z(X) & \hookrightarrow & \mathcal{N}i \smallsetminus z^0(X^0) \\ \downarrow & & \downarrow \\ \mathcal{N}i \smallsetminus z^1(X^1) & \hookrightarrow & \mathcal{N}i \end{array}$$

we conclude there exists a diagram in $\mathbf{SH}(S)$:

$$\begin{array}{ccc} V/V \smallsetminus i(X) & \xrightarrow{\cong} & \big(V/V \smallsetminus i^0(X^0)\big) \times \big(V/V \smallsetminus i^1(X^1)\big) \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{Bl}(i)/\mathbf{Bl}(i) \smallsetminus y(\mathbf{A}^1_X) & & \big(\mathbf{Bl}(i)/\mathbf{Bl}(i) \smallsetminus y^0(\mathbf{A}^1_{X^0})\big) \times \big(\mathbf{Bl}(i)/\mathbf{Bl}(i) \smallsetminus y^1(\mathbf{A}^1_{X^1})\big) \\ \cong \uparrow & & \uparrow \cong \\ \mathbf{Th}(\mathcal{N}i) & \xrightarrow{\cong} & \mathbf{Th}(\mathcal{N}i^0) \times \mathbf{Th}(\mathcal{N}i^1) \end{array}$$

The left vertical isomorphisms form part of the zig-zag of isomorphisms obtained from the homotopy purity theorem and the isomorphism between the Thom spaces is inverse to the canonical map

$$\mathbf{Th}(\mathcal{N}i) \xleftarrow{\nabla} \mathbf{Th}(\mathcal{N}i) \vee \mathbf{Th}(\mathcal{N}i) \leftarrow \mathbf{Th}(\mathcal{N}i^0) \vee \mathbf{Th}(\mathcal{N}i^1) \xrightarrow{\cong} \mathbf{Th}(\mathcal{N}i^0) \times \mathbf{Th}(\mathcal{N}i^1).$$

It remains to check that the map in $\mathbf{SH}(S)$ induced by the composition

$$\big(V/V \smallsetminus i^0(X^0)\big) \xrightarrow{\cong} \big(\mathbf{Bl}(i)/\mathbf{Bl}(i) \smallsetminus y^0(\mathbf{A}^1_{X^0})\big) \xleftarrow{\cong} \mathbf{Th}(\mathcal{N}i^0)$$

coincides with

$$\big(V/V \smallsetminus i^0(X^0)\big) \xrightarrow{\cong} \big(\mathbf{Bl}(i^0)/\mathbf{Bl}(i^0) \smallsetminus y^0(\mathbf{A}^1_{X^0})\big) \xleftarrow{\cong} \mathbf{Th}(\mathcal{N}i^0).$$

Now since X is a disjoint union of two closed subschemes the blow-up $\mathbf{Bl}(i)$ can be formed by successively blowing up X^0 and X^1. This furnishes a map $b\colon \mathbf{Bl}(i) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{Bl}(i^0)$ and diagrams in $\mathbf{SH}(S)$:

$$\begin{array}{ccccc} V & \hookrightarrow\!\!\!\!\!\rightarrow & \mathbf{Bl}(i) & \hookrightarrow & \mathcal{N}i \\ \| & & \downarrow{\scriptstyle b} & & \uparrow \\ V & \hookrightarrow\!\!\!\!\!\rightarrow & \mathbf{Bl}(i^0) & \hookrightarrow & \mathcal{N}i^0 \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{Bl}(i) \smallsetminus y^0(\mathbf{A}^1_{X^0}) & \hookrightarrow\!\!\circ\!\!\rightarrow & \mathbf{Bl}(i) \\ \downarrow & & \downarrow{\scriptstyle b} \\ \mathbf{Bl}(i^0) \smallsetminus y^0(\mathbf{A}^1_{X^0}) & \hookrightarrow\!\!\circ\!\!\rightarrow & \mathbf{Bl}(i^0) \end{array}$$

The result follows. □

Corollary 2.10: *Assumptions being as in Lemma 2.9, then every linear trivialization of $f\colon X \longrightarrow\!\!\!\!\!\rightarrow Y$ induces linear trivializations of f^0 and f^m and a diagram in $\mathbf{SH}(S)$:*

$$\begin{array}{ccc} Y_+ \wedge T^d & \xrightarrow{\big((i^0,p,\theta^0,\rho)^!,(i^1,p,\theta^1,\rho)^!\big)} & X^0_+ \vee X^1_+ \\ {\scriptstyle (i,p,\theta,\rho)^!}\downarrow & & \downarrow \\ X_+ & \xleftarrow{\qquad\qquad\nabla\qquad\qquad} & X_+ \vee X_+ \end{array}$$

Proof. This follows from Lemma 2.9. □

2.3 An alternate approach

Every finite étale map $f\colon X \longrightarrow\!\!\!\!\!\rightarrow Y$ in Sm_S induces maps $f_+\colon X_+ \longrightarrow \mathbf{1}_Y$ in $\mathbf{MSS}_Y$ and dually $\mathcal{D}_Y(f_+)\colon \mathcal{D}_Y(Y_+) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{D}_Y(X_+)$ in $\mathbf{SH}(Y)$ by applying the Spanier-Whitehead duality functor $\mathcal{D}_Y = \underline{\mathrm{Hom}}(-,\mathbf{1}_Y)$. The pullback functor $y^*\colon \mathbf{MSS}_S \longrightarrow\!\!\!\!\!\rightarrow \mathbf{MSS}_Y$ of the smooth map $y\colon Y \longrightarrow\!\!\!\!\!\rightarrow S$ has a left adjoint $y_\sharp\colon \mathbf{MSS}_Y \longrightarrow\!\!\!\!\!\rightarrow \mathbf{MSS}_S$ defined by sending $(W \longrightarrow\!\!\!\!\!\rightarrow Y)_+$ to $(W \longrightarrow\!\!\!\!\!\rightarrow Y \xrightarrow{y} S)_+$. The adjunction $(y_\sharp, y^*)$ is a Quillen pair by Lemma 4.1. We also write $(y_\sharp, y^*)$ for the total derived adjoint functor pair.

Definition 2.11: The duality transfer of a finite étale map $f\colon X \longrightarrow\!\!\!\!\!\rightarrow Y$ in Sm_S is the naturally induced map

$$y_\sharp\big(\mathcal{D}_Y(f_+)\big)\colon y_\sharp\big(\mathcal{D}_Y(Y_+)\big) \longrightarrow\!\!\!\!\!\rightarrow y_\sharp\big(\mathcal{D}_Y(X_+)\big)$$

in $\mathbf{SH}(S)$.

If $f\colon X \longrightarrow Y$ is a smooth projective map in Sm_S, let $\mathbf{Th}(\mathcal{T}f)$ denote the suspension spectrum of the Thom space of the tangent bundle of f and $\mathcal{D}_Y\big(\mathbf{Th}(\mathcal{T}f)\big)$ its dual in $\mathbf{MSS}_X$. From [9, Appendix] it follows there is an isomorphism in $\mathbf{SH}(Y)$

$$\mathcal{D}_Y(X_+) \cong f_\sharp\Big(\mathcal{D}_Y\big(\mathbf{Th}(\mathcal{T}f)\big)\Big). \tag{5}$$

If in addition f is étale, its tangent bundle $p\colon \mathcal{T}f \longrightarrow X$ has rank zero and using (5) we get an identification $\mathcal{D}_Y(X_+) \cong X_+$. Hence when $f = \mathrm{id}_Y$, the canonical isomorphism $\mathcal{D}_Y(Y_+) \cong Y_+$ in $\mathbf{SH}(Y)$ implies the following result.

Lemma 2.12: *The duality transfer of $f = \mathrm{id}_Y$ is the identity map id_{Y_+} in $\mathbf{SH}(S)$.*

In addition, we claim duality transfer maps satisfy the exact same type of properties as the transfer maps considered in Section 2.1. To state compatibility with respect to base change along a map $i\colon Z \longrightarrow Y$ in Sm_S, observe that for every dualizable motivic symmetric spectrum E over Y applying [5, 3.1] to the strict symmetric monoidal functor $i^*\colon \mathbf{SH}(Y) \longrightarrow \mathbf{SH}(Z)$ shows there is a canonical isomorphism

$$i^*\big(\mathcal{D}_Y(E)\big) \cong \mathcal{D}_Z(i^*E). \tag{6}$$

In particular, there is a canonical morphism $z_\sharp\big(\mathcal{D}_Z(i^*E)\big) \longrightarrow y_\sharp\big(\mathcal{D}_Y(E)\big)$ adjoint to the composition

$$\mathcal{D}_Z(i^*E) \xrightarrow{\cong} i^*\big(\mathcal{D}_Y(E)\big) \longrightarrow i^*y^*y_\sharp\big(\mathcal{D}_Y(E)\big) \xrightarrow{\cong} z^*y_\sharp\big(\mathcal{D}_Y(E)\big). \tag{7}$$

Lemma 2.13: *Every pullback diagram in Sm_S where $f\colon X \longrightarrow Y$ is a finite étale map*

$$\begin{array}{ccc} W & \longrightarrow & X \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

induces a diagram between duality transfer maps in $\mathbf{SH}(S)$:

$$\begin{array}{ccc} z_\sharp\big(\mathcal{D}_Z(W_+)\big) & \longrightarrow & y_\sharp\big(\mathcal{D}_Y(X_+)\big) \\ {\scriptstyle z_\sharp(\mathcal{D}_Z(g_+))}\uparrow & & \uparrow{\scriptstyle y_\sharp(\mathcal{D}_Y(f_+))} \\ z_\sharp\big(\mathcal{D}_Z(Z_+)\big) & \longrightarrow & y_\sharp\big(\mathcal{D}_Y(Y_+)\big) \end{array}$$

Proof. It suffices to consider the adjoint diagram:

$$\begin{array}{ccc} \mathcal{D}_Z(W_+) & \rightarrowtail & z^* y_\sharp\big(\mathcal{D}_Y(X_+)\big) \\ {\scriptstyle \mathcal{D}_Z(g_+)}\uparrow & & \uparrow{\scriptstyle z^* y_\sharp(\mathcal{D}_Y(f_+))} \\ \mathcal{D}_Z(Z_+) & \longrightarrow & z^* y_\sharp\big(\mathcal{D}_Y(Y_+)\big) \end{array}$$

Naturality of the isomorphism (6) and the composition (7) shows that it commutes. □

In the situation of Lemma 2.13, the tangent bundle of g is isomorphic to the pullback of the tangent bundle of f. A tedious check reveals that the identifications obtained from (5) are compatible under pullbacks.

Lemma 2.14: *Suppose $X = X^0 \coprod X^1$ is the disjoint union of finite étale Y-schemes $f^0\colon X^0 \longrightarrow Y$ and $f^1 \rightarrowtail X \longrightarrow Y$. Define $f \equiv f^0 \coprod f^1$. Then there is a diagram in $\mathbf{SH}(S)$ where the right vertical map is the canonical isomorphism:*

$$\begin{array}{ccc} y_\sharp(\mathcal{D}_Y(Y_+)) & \xrightarrow{\quad y_\sharp\mathcal{D}_Y(f_+) \quad} & y_\sharp(\mathcal{D}_Y(X_+)) \\ {\scriptstyle \triangle}\downarrow & & \downarrow{\scriptstyle \cong} \\ y_\sharp(\mathcal{D}_Y(Y_+)) \times y_\sharp(\mathcal{D}_Y(Y_+)) & \xrightarrow{y_\sharp\mathcal{D}_Y(f^0_+) \times y_\sharp\mathcal{D}_Y(f^1_+)} & y_\sharp(\mathcal{D}_Y(X^0_+)) \times y_\sharp(\mathcal{D}(X^1_+)) \end{array}$$

Proof. We may assume $Y = S$, in which case the lemma follows from the fact that $\mathcal{D}_S$ preserves finite products. □

Remark 2.15: Lemma 2.9, Corollary 2.10 and Lemma 2.14 generalize immediately to the case when X is a finite disjoint union of schemes.

3 Moore spectra

Let $n > 1$ be an integer and $n\colon \mathbf{1}_S \longrightarrow \mathbf{1}_S$ an automorphism of the motivic sphere spectrum representing multiplication by n on the unit in $\mathbf{SH}(S)$. In effect, $\mathbf{SH}(S)$ is an additive category since the natural map $E \vee F \longrightarrow E \times F$ is a weak equivalence for all motivic symmetric spectra E and F. The sum of $\alpha, \beta\colon E \longrightarrow F$ is the composite map

$$E \xrightarrow{\triangle} E \times E \xrightarrow{\cong} E \vee E \xrightarrow{\alpha\vee\beta} F.$$

The mod-n Moore spectrum $\mathbf{1}^n_S$ is defined by the homotopy cofiber sequence

$$\mathbf{1}_S \xrightarrow{n} \mathbf{1}_S \xrightarrow{\delta} \mathbf{1}^n_S \xrightarrow{\epsilon} S^1 \wedge \mathbf{1}_S. \tag{8}$$

The maps $\mathrm{Hom}_{\mathbf{SH}(S)}(n, E)$ and $\mathrm{Hom}_{\mathbf{SH}(S)}(E, n)$ are multiplication by n for every motivic spectrum E. Thus, by applying $\mathrm{Hom}_{\mathbf{SH}(S)}(-, \mathbf{1}^n_S)$ to (8), we conclude that $\delta^*(n\,\mathrm{id}_{\mathbf{1}^n_S}) = 0$. Again by exactness there exists a map $\alpha\colon S^1 \wedge \mathbf{1} \longrightarrow \mathbf{1}^n_S$ such that $\epsilon^*(\alpha) = n\,\mathrm{id}_{\mathbf{1}^n_S}$. Clearly the element $n\alpha$ is in the image of n^* and hence $n^2\,\mathrm{id}_{\mathbf{1}^n_S} = n\epsilon^*(\alpha) = \epsilon^*(n\alpha) = 0$. It follows that $\mathrm{Hom}_{\mathbf{SH}(S)}(E, \mathbf{1}^n_S \wedge F)$ and $\mathrm{Hom}_{\mathbf{SH}(S)}(\mathbf{1}^n_S \wedge E, F)$ are $\mathbb{Z}/n^2$-modules for all E and F.

Remark 3.1: One way of constructing $\mathbf{1}^n_S$ is to take the image of the topological Moore spectrum under the canonical additive functor $\mathbf{SH} \longrightarrow \mathbf{SH}(S)$. With this choice there is an isomorphism $f^*\mathbf{1}^n_R \cong \mathbf{1}^n_S$ in $\mathbf{SH}(S)$ for every map $f\colon S \longrightarrow R$ of base schemes.

Remark 3.2: It is also of interest to consider Moore spectra for subrings $\mathbb{Z}[J^{-1}] \hookrightarrow \mathbb{Q}$, where J is a set of prime numbers.

Remark 3.3: In general, the group $\pi_{0,0}\mathbf{1}_S = \mathrm{Hom}_{\mathbf{SH}(S)}(\mathbf{1}_S, \mathbf{1}_S)$ contains more elements than just integers. For example, if S is the spectrum of a perfect field k of characteristic different from 2, one may consider Moore spectra with respect to any element in the Grothendieck-Witt ring of quadratic forms over k [11].

Remark 3.4: If multiplication by n is injective on $\pi_{0,0}\mathbf{1}_S$ and $\pi_{1,0}\mathbf{1}_S/n\pi_{1,0}\mathbf{1}_S$ consists of elements of order prime to n, then $\mathrm{Hom}_{\mathbf{SH}(S)}(\mathbf{1}^n_S \wedge E, F)$ and $\mathrm{Hom}_{\mathbf{SH}(S)}(E, \mathbf{1}^n_S \wedge F)$ are $\mathbb{Z}/n$-modules for all E and F. According to [11], the first condition holds for algebraically closed fields and for real closed fields provided n is odd. If k is a subfield of the complex numbers, taking complex points implies $\pi_{1,0}\mathbf{1}_k$ contains $\pi_1 S^0 = \mathbb{Z}/2$ as a direct summand. For algebraically closed fields of characteristic zero, it seems reasonable to expect that $\pi_{1,0}\mathbf{1}_k$ is isomorphic to $\pi_1 S^0$.

A map $\alpha\colon E \longrightarrow F$ in $\mathbf{MSS}_S$ is an $\mathbf{1}^n_S$-equivalence if $\mathrm{id} \wedge \alpha\colon \mathbf{1}^n_S \wedge E \longrightarrow \mathbf{1}^n_S \wedge F$ is a stable equivalence. In Appendix A we show the classes of $\mathbf{1}^n_S$-equivalences and ordinary cofibrations form a model structure $\mathbf{MSS}^n_S$ on the category of motivic symmetric spectra. The identity is then a left Quillen functor $\mathbf{MSS}_S \longrightarrow \mathbf{MSS}^n_S$. Let $\mathbf{SH}(S)^n$ denote the corresponding mod-n motivic stable homotopy category.

Example 3.5: Following Bousfield [3] we shall construct a fibrant replacement functor in the mod-n model structure $\mathbf{MSS}^n_S$.

First, for every motivic symmetric spectrum E we note there is a tower

$$\mathbf{1}_S^n \wedge E \prec\!\!\leftarrow \mathbf{1}_S^{n^2} \wedge E \prec\!\!\leftarrow \mathbf{1}_S^{n^3} \wedge E \prec\!\!\leftarrow \cdots . \tag{9}$$

In effect, let $k > 0$ be an integer and consider the diagram:

$$\begin{array}{ccccccc}
\mathbf{1}_S & \xrightarrow{n} & \mathbf{1}_S & \longrightarrow & \mathbf{1}_S^n & \longrightarrow & S^1 \wedge \mathbf{1}_S \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle n^k} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathbf{1}_S & \xrightarrow{n^{k+1}} & \mathbf{1}_S & \longrightarrow & \mathbf{1}_S^{n^{k+1}} & \longrightarrow & S^1 \wedge \mathbf{1}_S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{1}_S^{n^k} & \longrightarrow & F & \longrightarrow & *
\end{array} \tag{10}$$

The upper and middle rows and all the columns are distinguished triangles in $\mathbf{SH}(S)$. Hence the lower row is a distinguished triangle, and there exist maps $\mathbf{1}_S^{n^{k+1}} \longrightarrow \mathbf{1}_S^{n^k}$ which are compatible with the unit. Smashing with E in $\mathbf{MSS}_S$ yields the tower (9). We claim that taking its homotopy limit gives a fibrant replacement functor in $\mathbf{MSS}_S^n$. Applying the Spanier-Whitehead duality functor $\mathcal{D} = \underline{\mathrm{Hom}}(-, \mathbf{1}_S)$ gives an identification

$$\mathcal{D}(\mathbf{1}_S^n \prec\!\!\leftarrow \mathbf{1}_S^{n^2} \prec\!\!\leftarrow \mathbf{1}_S^{n^3} \prec\!\!\leftarrow \cdots) = (S^{-1,0} \wedge \mathbf{1}_S^n \rightarrow S^{-1,0} \wedge \mathbf{1}_S^{n^2} \rightarrow \cdots). \tag{11}$$

This uses the canonical isomorphism $\mathcal{D}(\mathbf{1}_S) = \mathbf{1}_S$ which implies $\mathcal{D}(\mathbf{1}_S^n) = S^{-1,0} \wedge \mathbf{1}_S^n$ is the desuspension of the mod-n Moore spectrum. Let $S^{-1,0} \wedge \mathbf{1}_S^{n^\infty}$ denote the homotopy colimit of (11), so that the homotopy limit of (9) is isomorphic to $\underline{\mathrm{Hom}}(S^{-1,0} \wedge \mathbf{1}_S^{n^\infty}, E)$. If $\mathbf{1}_S^n \wedge F$ is contractible, or equivalently if F is $\mathbf{1}_S^n$-acyclic in the sense of Definition A.6, it follows that $\mathbf{1}_S^{n^\infty} \wedge F$ is a homotopy colimit of contractible objects. Thus for every $\mathbf{1}_S^n$-acyclic spectrum F we get

$$\mathrm{Hom}_{\mathbf{SH}(S)}\big(F, \underline{\mathrm{Hom}}(S^{-1,0} \wedge \mathbf{1}_S^{n^\infty}, E)\big) = \mathrm{Hom}_{\mathbf{SH}(S)}(S^{-1,0} \wedge \mathbf{1}_S^{n^\infty} \wedge F, E) = 0.$$

In homotopical algebraic terms the internal hom $\underline{\mathrm{Hom}}(S^{-1,0} \wedge \mathbf{1}_S^{n^\infty}, E)$ is called $\mathbf{1}_S^n$-local. Applying the internal hom functor $\underline{\mathrm{Hom}}(-, E)$ to the distinguished triangle

$$S^{-1,0} \wedge \mathbf{1}_S^{n^\infty} \longrightarrow \mathbf{1}_S \longrightarrow \mathbf{1}_S[n^{-1}] \longrightarrow \mathbf{1}_S^{n^\infty}$$

we get an induced distinguished triangle

$$\underline{\mathrm{Hom}}(S^{-1,0} \wedge \mathbf{1}_S^{n^\infty}, E) \prec\!\!\leftarrow \underline{\mathrm{Hom}}(\mathbf{1}_S, E) = E \prec\!\!\leftarrow \underline{\mathrm{Hom}}(\mathbf{1}_S[n^{-1}], E) \prec\!\!\leftarrow \underline{\mathrm{Hom}}(\mathbf{1}_S^{n^\infty}, E).$$

It follows that $\mathbf{1}^n_S \wedge E[n^{-1}]$ is trivial in $\mathbf{SH}(S)$ and the map $E \to \underline{\mathrm{Hom}}(S^{-1,0} \wedge \mathbf{1}^{n^\infty}_S, E)$ is an $\mathbf{1}^n_S$-equivalence with a fibrant target in the mod-n model structure.

The mod-n Moore spectrum $\mathbf{1}^n_S$ is a wedge of mod-ℓ^ν Moore spectra according to the primary factors ℓ of n. In what follows we denote the explicit fibrant replacements in the mod-ℓ model structure, a.k.a. ℓ-adic completions, by $E^{\hat{}}_\ell \equiv \underline{\mathrm{Hom}}(S^{-1,0} \wedge \mathbf{1}^{\ell^\infty}_S, E)$. We note there are short exact sequences of bigraded motivic stable homotopy groups

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/\ell^\infty, \pi_{p,q}E) \longrightarrow \pi_{p,q}E^{\hat{}}_\ell \longrightarrow \mathrm{Hom}(\mathbb{Z}/\ell^\infty, \pi_{p-1,q}E) \longrightarrow 0.$$

4 Motivic rigidity

Let $f\colon S \longrightarrow R$ be a map of base schemes. By base change there is a strict symmetric monoidal left Quillen functor $f^*\colon \mathbf{MSS}_R \longrightarrow \mathbf{MSS}_S$. It descends to a left Quillen functor f^* on the mod-ℓ model structures since $f^*(\mathbf{1}^\ell_R) \longrightarrow \mathbf{1}^\ell_S$ is a stable equivalence. If f is smooth, then since every motivic space is a colimit of representable ones, setting

$$f_\sharp(X \longrightarrow S) \equiv (X \longrightarrow S \xrightarrow{f} R)$$

defines an op-lax symmetric monoidal functor and an induced adjoint functor pair:

$$f_\sharp\colon \mathbf{MSS}_S \rightleftarrows \mathbf{MSS}_R\colon f^*$$

Let $\varepsilon\colon f_\sharp f^* \longrightarrow \mathrm{Id}_{\mathbf{MSS}_R}$ denote the counit of the adjunction. The natural isomorphism $f_\sharp(A \wedge f^*B) \longrightarrow f_\sharp(A) \wedge B$, see for example [12, 3.1.23] for the motivic space version, implies that $f_\sharp$ preserves stable equivalences and hence it is a left Quillen functor.

The next lemma sets up the Quillen adjoint pair which figures in the proof of the motivic rigidity theorem.

Lemma 4.1: *If S is a filtered limit of smooth schemes over R with affine transition maps, then there is an induced Quillen adjoint pair:*

$$f_\sharp\colon \mathbf{MSS}_S \rightleftarrows \mathbf{MSS}_R\colon f^*$$

In particular, the total left derived functor of f^* has a left adjoint since it maps by a natural isomorphism to the right derived functor of f^*.

Proof. The adjunction follows from the induced adjunction on the level of motivic spaces in [12, 3.1.24], where it is proved that f^* preserves weak equivalences. Working unstably, to get a Quillen pair it suffices that $f_\sharp$ preserves fibrations between fibrant motivic spaces. Using the set J' of acyclic cofibrations in [4, 2.14] which detects fibrations between fibrant motivic spaces relative to R, this follows provided f^* sends every object of J' to an acyclic cofibration. This holds because of the characterizing property $f^*(X) = S \times_R X$. It is now straightforward to lift the Quillen adjunction to the level of motivic symmetric spectra because the unit and counit of the adjunction between motivic spaces extend to the setup of motivic symmetric spectra. □

Let k be a field. If the real spectrum of k is non-empty, taking real points shows the map $\phi(k)\colon \mathbf{GL}_n(k) \longrightarrow \mathrm{Aut}_{\mathbf{SH}(k)}(k_+)$ is non-trivial since, for example, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{O}(2)$$

induces a degree -1 map on S^1. However, we have:

Lemma 4.2: *If every unit in k is a square, then $\phi(k)$ is the trivial map.*

Proof. It follows that $\phi(k)$ factors through $k^\times$ by comparing (2) and the short exact sequence $0 \longrightarrow \mathbf{SL}_n(k) \longrightarrow \mathbf{GL}_n(k) \longrightarrow k^\times \longrightarrow 0$. In homogeneous coordinates on $\mathbf{P}^1$ the value of $\phi(k)$ at a square $u^2 \in k^\times$ is given by the matrix

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \mathbf{SL}_2(k).$$

As noted in Remark 2.6, every element of $\mathbf{SL}_2(k)$ is a product of elementary matrices which are contractible via linear homotopies. Thus $\phi(k)$ factors through $k^\times/(k^\times)^2$ which is trivial by assumption. □

Corollary 4.3: *Suppose every element in $k^\times$ is a square. Then $(i, p, \theta, \rho)^!$ is the identity for every linear trivialization of the identity map on k.*

Proof. Since every linear trivialization induces the identity map according to Lemma 4.2, it suffices to note that sections of $\mathbf{A}^n$ map to the zero section via linear automorphisms of $\mathbf{A}^n$ because the base scheme is a field. □

Remark 4.4: If every element of a field k is a square and the exponential characteristic $\mathrm{char}(k) \neq 2$, then the Grothendieck-Witt ring $GW(k)$ of k is isomorphic to the integers. If k is perfect and $\mathrm{char}(k) \neq 2$, then $\mathrm{Aut}_{\mathbf{SH}(k)}(k_+)$ is isomorphic to $GW(k)$ by [11].

If C is a curve over a field k then the free abelian group of divisors $\mathrm{Div}(C)$ is generated by closed embeddings $\{x\} \hookrightarrow C$, where the residue field $k(x)$ of x is a finite extension of k. By considering the induced maps in $\mathbf{SH}(k)$ for an algebraically closed field k and extending by linearity we get a group homomorphism

$$\Phi_C \colon \mathrm{Div}(C) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(k)}(k_+, C_+).$$

Remark 4.5: If k is an arbitrary perfect field, using duality transfers for the finite étale maps $\mathrm{Spec}\big(k(x)\big) \longrightarrow \mathrm{Spec}(k)$ we get a map $\mathrm{Div}(C) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(k)}(k_+, C_+)$ which factors through $\mathrm{Hom}_{\mathbf{SH}(k)}\big(k(x)_+, C_+\big)$.

Theorem 4.6: *Suppose k is an algebraically closed field, C is an affine curve in Sm_k and choose a projective completion $j \colon C \hookrightarrow \overline{C}$ with finite closed complement $C_\infty \hookrightarrow \overline{C}$. Then Φ_C factors canonically through the relative Picard group of $\overline{C}$ and C_∞ as in*

$$\Phi_C \colon \mathrm{Div}(C) \longrightarrow \mathrm{Pic}(\overline{C}, C_\infty) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(k)}(k_+, C_+).$$

Proof. The assumption on k implies that $\mathrm{Pic}(\overline{C}, C_\infty)$ is generated by divisors which are smooth and unramified over C. Thus it suffices to show Φ_C vanishes on principal divisors $\mathrm{div}(f) = f^{-1}(0) - f^{-1}(\infty)$, where $f \in k(\overline{C})^\times$ induces a dominant map $f \colon \overline{C} \longrightarrow \mathbf{P}^1$ which is unramified over 0 and ∞, and $f(C_\infty) \equiv 1$. Set $D^0 \equiv f^{-1}(0)$ and $D^\infty \equiv f^{-1}(\infty)$. The subset $f^{-1}\big(\mathbf{P}^1 \smallsetminus \{1\}\big)$ on $\overline{C}$ defines an open affine subscheme $j \colon U \hookrightarrow C$ such that $D^0, D^\infty \subseteq U$ and $f \circ j$ factors through $\mathbf{A}^1 = \mathbf{P}^1 \smallsetminus \{1\}$ via a finite affine map $\phi \colon U \longrightarrow \mathbf{A}^1$. Choose an open subset $U' \hookrightarrow U$ containing D^0 and D^∞ such that U' has a trivial tangent bundle. Since U' is affine, there is a closed embedding $U' \hookrightarrow \mathbf{A}^n$. The composite map

$$U' \overset{\Gamma(\phi')}{\hookrightarrow} U' \times \mathbf{A}^1 \hookrightarrow \mathbf{A}^n \times \mathbf{A}^1 \overset{\mathrm{pr}}{\longrightarrow} \mathbf{A}^1$$

is a linearization (i, pr) of $\phi' \colon U' \hookrightarrow U \overset{\phi}{\longrightarrow} \mathbf{A}^1$. Note that the short exact sequence of vector bundles $0 \longrightarrow \mathcal{T}U' \longrightarrow i^*\mathcal{T}\mathbf{A}^{n+1} \longrightarrow \mathcal{N}i \longrightarrow 0$ splits because U' is affine. Using this setup we deduce that $\mathcal{N}i$ is a stably trivial vector bundle over the smooth curve U', so by the cancellation theorem [2, IV 3.5] it is isomorphic to a trivial bundle. This shows that ϕ' is linearly trivial.

The normal bundle $\mathcal{N}i$ restricts to the respective normal bundles on the disjoint unions D^0 and D^∞ of closed points on U' and likewise for any linear trivialization of ϕ'. Thus, by Corollary 2.8, the points 0 and ∞ on $\mathbf{P}^1$ induce a diagram in $\mathbf{SH}(k)$:

$$\begin{array}{ccccc} & \overset{0}{} & & \overset{\infty}{} & \\ \mathrm{Spec}(k)_+ & \overset{\subset}{\longrightarrow\!\!\!\!+} & (\mathbf{P}^1 \smallsetminus \{1\})_+ & \overset{\supset}{+\!\!\!\!\longleftarrow} & \mathrm{Spec}(k)_+ \\ \phi_0^!\downarrow & & (\phi')^!\downarrow & & \phi_\infty^!\downarrow \\ D^0_+ & \overset{\subset}{\longrightarrow\!\!\!\!+} & U'_+ & \overset{\supset}{+\!\!\!\!\longleftarrow} & D^\infty_+ \end{array} \tag{12}$$

By Corollary 4.3 the left and right vertical transfer maps in (12) are independent of the linear trivialization. Corollary 2.10 implies the composite map

$$\mathrm{Spec}(k)_+ \xrightarrow{\phi_0^!} D^0_+ \overset{\subset}{\longrightarrow\!\!\!\!+} U'_+ \overset{\subset}{\longrightarrow\!\!\!\!\circ} C_+$$

coincides with the map $\Phi_C(D^0)\colon \mathrm{Spec}(k)_+ \longrightarrow C_+$ in $\mathbf{SH}(k)$, and similarly for ∞. This shows that (12) induces an $\mathbf{A}^1$-homotopy between $\Phi_C(D^0)$ and $\Phi_C(D^\infty)$. □

Remark 4.7: By reference to Lemmas 2.12, 2.13 and 2.14 the argument for Theorem 4.6 goes through using duality transfer maps provided $\phi\colon U \longrightarrow \mathbf{A}^1$ is finite étale.

Corollary 4.8: *Suppose $n > 1$ is prime to the exponential characteristic* $\mathrm{char}(k)$*, C is an affine curve in* Sm_k *and* $A\colon \mathbf{SH}(k) \longrightarrow \mathcal{C}$ *is an additive functor such that* $\mathrm{Hom}_{\mathcal{C}}\big(A(k_+), A(C_+)\big)$ *is n-torsion. Then for all divisors D and D' on C of the same degree, we have*

$$A\big(\Phi_C(D)\big) = A\big(\Phi_C(D')\big)\colon A(k_+) \longrightarrow A(C_+).$$

In particular, the composite map

$$\mathrm{Pic}(\overline{C}, C_\infty) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(k)}(k_+, C_+) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\big(A(k_+), A(C_+)\big)$$

factors through the degree map $\mathrm{Pic}(\overline{C}, C_\infty) \longrightarrow \mathbb{Z}$ *and for every divisor D of degree zero the map $A\big(\Phi_C(D)\big)$ is trivial.*

Proof. The kernel $\mathrm{Pic}^0(\overline{C}, C_\infty)$ of the degree map is n-divisible since multiplication by n is surjective on $k^\times$ while on the Jacobian of $\overline{C}$ multiplication by n is a finite map between irreducible varieties of the same dimension and hence a surjection on k-points. Now since the composite map sends n-divisible elements to zero in $\mathcal{C}$, we are done. □

Theorem 4.9: *If X is an affine scheme in Sm_k and $p_0, p_1 \in X(k)$ are k-rational points, then the induced maps $p_0^*, p_1^*\colon A(k_+) \longrightarrow A(X_+)$ in $\mathcal{C}$ coincide.*

Proof. Follows from Corollary 4.8 because p_0 and p_1 can be connected by an irreducible smooth affine curve $C \subset X$ [13, pg. 56]. □

Let K/k be an extension of algebraically closed fields of exponential characteristic prime to a fixed prime number ℓ and let f be the corresponding map of affine schemes. Define $F/\ell^\nu \equiv \mathbf{1}_k^{\ell^\nu} \wedge F$ for a motivic symmetric spectrum F in $\mathbf{MSS}_k$.

Theorem 4.10: *For motivic symmetric spectra E and F there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{SH}(k)}\big(\varepsilon(E), F/\ell^\nu\big)\colon \mathrm{Hom}_{\mathbf{SH}(k)}(E, F/\ell^\nu) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{SH}(k)}\big(f_\sharp f^*(E), F/\ell^\nu\big).$$

Proof. The main input in the proof is Theorem 4.9 applied to the torsion group valued additive functor

$$E \longmapsto \mathrm{Hom}_{\mathbf{SH}(K)}\big(E, f^*(F/\ell^\nu)\big). \tag{13}$$

By reducing to generators of the triangulated motivic stable homotopy category $\mathbf{SH}(k)$ we may assume E is the suspension spectrum X_+ of an affine scheme in Sm_k. First we consider the case $X = \mathrm{Spec}(k)$. Since K is a colimit of algebraically closed subfields of finite transcendence degree over k, we may assume K/k has transcendence degree one. Hence K is a filtered colimit of smooth finitely generated k-subalgebras R and there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbf{SH}(k)}\big(f_\sharp f^*\mathbf{1}_k, F/\ell^\nu\big) = \operatorname*{colim}_{k\subset R\subset K} \mathrm{Hom}_{\mathbf{SH}(k)}\big(\mathrm{Spec}(R)_+, F/\ell^\nu\big).$$

Injectivity of $\mathrm{Hom}_{\mathbf{SH}(k)}\big(\varepsilon(\mathbf{1}_k), F/\ell^\nu\big)$ follows since the Nullstellensatz shows there exists a map $\phi\colon R \longrightarrow k$ which restricts to the identity on k. To prove surjectivity it suffices to show that the map $\mathrm{Hom}_{\mathbf{SH}(k)}\big((R \subset K)_+, F/\ell^\nu\big)$ factors through $\mathrm{Hom}_{\mathbf{SH}(k)}\big(\varepsilon(k_+), F/\ell^\nu\big)$, i.e. there is an equality between the naturally induced maps

$$(R \subset K)_+^*, (k \subset K)_+^* \circ \phi_+^*\colon \mathrm{Hom}_{\mathbf{SH}(k)}\big(\mathrm{Spec}(R)_+, F/\ell^\nu\big) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(k)}\big(f_\sharp f^*\mathbf{1}_k, F/\ell^\nu\big).$$

Every k-algebra homomorphism $\psi\colon R \longrightarrow K$ factors through $R\otimes_k K$ via $r \longmapsto r\otimes 1$ and $r \otimes x \longmapsto \psi(r)x$; in particular, the k-algebra homomorphisms in question correspond to K-rational points p_0 and p_1 on the affine curve $\mathrm{Spec}(R \otimes_k K)$ in Sm_K and there are induced maps

$$p_0^*, p_1^*\colon \mathrm{Hom}_{\mathbf{SH}(K)}\big(f^*\mathbf{1}_k, f^*(F/\ell^\nu)\big) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(K)}\big(f^*\,\mathrm{Spec}(R)_+, f^*(F/\ell^\nu)\big).$$

Theorem 4.9 implies $p_0^* = p_1^*$ by inserting $E = \mathrm{Spec}(R)_+$ into (13). Combining this with the natural isomorphism

$$\mathrm{Hom}_{\mathbf{SH}(K)}\big(f^*X_+, f^*(F/\ell^\nu)\big) \cong \mathrm{Hom}_{\mathbf{SH}(k)}(f_\sharp f^* X_+, F/\ell^\nu)$$

for $X = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(R)$, and comparing with the map

$$\mathrm{Hom}_{\mathbf{SH}(k)}(\mathrm{Spec}(R)_+, F/\ell^\nu) \longrightarrow \mathrm{Hom}_{\mathbf{SH}(k)}(f_\sharp f^* \,\mathrm{Spec}(R)_+, F/\ell^\nu),$$

we conclude that

$$(R \subset K)_+^* = (k \subset K)_+^* \circ \phi_+^*.$$

The argument extends to all affine schemes in Sm_k by forming pullbacks. □

Remark 4.11: Note that $\mathrm{Hom}_{\mathbf{SH}(k)}\big(\varepsilon(E), F\big)$ is injective for every E and F.

Let $\mathbf{sSet}_{\mathbf{MSS}_k}(E, F)$ denote the function complex of maps from E to F in $\mathbf{MSS}_k$. Recall that an adjunction is called a reflection if its counit is a natural isomorphism. We are ready to prove the motivic rigidity theorem:

Theorem 4.12: *The total derived Quillen adjunction*

$$f_\sharp \colon \mathbf{SH}^\ell(K) \rightleftarrows \mathbf{SH}^\ell(k) \colon f^*$$

is a reflection.

Proof. We show that for motivic symmetric spectra E and F there is an isomorphism

$$\mathrm{Hom}_{\mathbf{SH}(k)^\ell}\big(\varepsilon(E), F\big) \colon \mathrm{Hom}_{\mathbf{SH}(k)^\ell}(E, F) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{SH}(k)^\ell}\big(f_\sharp f^*(E), F\big).$$

Theorem 4.10 implies that $\mathbf{sSet}_{\mathbf{MSS}_k}\big(\varepsilon(E), R(F/\ell^\nu)\big)$ is a weak equivalence for cofibrant motivic symmetric spectra E and F, and for every fibrant replacement R. The functor $\mathbf{sSet}_{\mathbf{MSS}_k}(E, -)$ commutes with homotopy limits because of the cofibrancy assumption. Since the ℓ-adic completion $F\hat{}_\ell$ of F is isomorphic to the homotopy limit of the diagram $\nu \longmapsto F/\ell^\nu$ by Example 3.5, it follows that $\mathbf{sSet}_{\mathbf{MSS}_k}\big(\varepsilon(E), F\hat{}_\ell\big)$ is a weak equivalence. In other terms, the map $\mathrm{Hom}_{\mathbf{SH}(k)^\ell}\big(\varepsilon(E), F\big)$ is an isomorphism for every E and F. □

Recall that an adjunction is a reflection if and only if its right adjoint is full and faithful, so that we deduce the motivic rigidity theorem stated in the introduction.

Let L be a motivic symmetric spectrum in $\mathbf{MSS}_k$. In the formulations of the next results we do not distinguish notationally between left and right modules.

Theorem 4.13: *If L/ℓ is a monoid in $\mathbf{MSS}_k$ the total derived Quillen adjunction of*

$$f_\sharp\colon f^*L/\ell - \mathbf{mod} \rightleftarrows L/\ell - \mathbf{mod}\colon f^*$$

is a reflection.

Proof. Example A.7 shows that the generators $L/\ell \wedge X_+$ of the homotopy category of $L/\ell - \mathbf{mod}$ are fibrant in the L-local model structure detailed in Appendix A. Applying Theorem 4.9 to the ℓ-torsion valued additive functor $E \longmapsto \mathrm{Hom}_{\mathbf{SH}(K)}\big(E, f^*(L/\ell \wedge F)\big)$, the proof runs in parallel with the argument for Theorem 4.12. □

The motivic Eilenberg-MacLane spectrum $\mathbf{MZ}_k$ in $\mathbf{MSS}_k$ satisfies the conditions in Theorem 4.13 and there is an isomorphism $f^*\mathbf{MZ}_k/\ell \xrightarrow{\cong} \mathbf{MZ}_K/\ell$ in $\mathbf{MSS}_K$.

Corollary 4.14: *The total right derived functor of*

$$f^*\colon \mathbf{MZ}_k/\ell - \mathbf{mod} \longrightarrow \mathbf{MZ}_K/\ell - \mathbf{mod}$$

is fully faithful.

Remark 4.15: For fields of characteristic zero, Corollary 4.14 implies rigidity for big categories of motives by [15] and hence for effective motives by Voevodsky's cancellation theorem [17], cf. [8]. There exists an analog of Corollary 4.14 for algebras over $\mathbf{MZ}_k/\ell$.

Since the assumption on L in Theorem 4.13 excludes several important examples of motivic symmetric spectra, see Remark 4.17, we note there is another closely related and more applicable rigidity theorem; using generators, the proof is a verbatim copy of the argument for Theorem 4.13.

Theorem 4.16: *If L/ℓ is a monoid in $\mathbf{SH}(k)$ there is a naturally induced reflection of categories of modules in motivic stable homotopy categories:*

$$\mathbf{SH}(K)\big(f^*L/\ell\big) - \mathbf{mod} \rightleftarrows \mathbf{SH}(k)\big(L/\ell\big) - \mathbf{mod}$$

Remark 4.17: Suppose L is a monoid in $\mathbf{MSS}_k$. It is not necessarily true that L/ℓ is a monoid in either $\mathbf{MSS}_k$ or in $\mathbf{SH}(k)$. If $\ell \geq 5$ it follows that L/ℓ is a monoid in $\mathbf{SH}(k)$ since then the mod-ℓ Moore spectrum acquires a homotopy associative and commutative multiplication. The fact that there is no monoid whose underlying spectrum provides a model for the mod-ℓ Moore spectrum is a pertaining source of technical fun in stable homotopy theory. It is not known whether Theorem 4.16 implies Corollary 4.14.

A Homological localization

The purpose of this appendix is to work out the homotopical foundation for a localization theory of motivic symmetric spectra. Our main result follows by adjusting arguments due to Bousfield [3] for spectra and Goerss-Jardine [7] for simplicial presheaves. Recall from [4] there exists a set J of acyclic cofibrations $j\colon \mathrm{d}j \xrightarrow{\sim} \mathrm{c}j$ with finitely presentable and cofibrant domains and codomains such that E is fibrant in $\mathbf{MSS}_S$ if and only if the map $E \longrightarrow *$ has the right lifting property with respect to J.

A.1 A fibrant replacement

Applying the small object argument to $E \longrightarrow *$ and J furnishes for any E a stably fibrant motivic symmetric spectrum $\mathcal{R}(E)$: Let $\mathcal{R}^1(E)$ be the pushout of

$$\bigvee_{\coprod_{j\in J}\mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j,E)} \mathrm{c}j \xleftarrow{\sim} \bigvee_{\coprod_{j\in J}\mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j,E)} \mathrm{d}j \longrightarrow E.$$

This construction is clearly natural in E, there is an acyclic cofibration $E \xrightarrow{\sim} \mathcal{R}^1(E)$ and a natural transformation $\rho^1\colon \mathrm{Id}_{\mathbf{MSS}_S} \longrightarrow \mathcal{R}^1$. Let $\mathcal{R}(E)$ denote the colimit of

$$E \xrightarrow[\sim]{\rho^1(E)} \mathcal{R}^1(E) \xrightarrow[\sim]{\rho^1(\mathcal{R}^1(E))} \mathcal{R}^1\big(\mathcal{R}^1(E)\big) \xrightarrow[\sim]{} \dots .$$

There is an induced natural transformation $\rho\colon \mathrm{Id}_{\mathbf{MSS}_S} \longrightarrow \mathcal{R}$. We shall identify Sm_S (up to equivalence) with a small skeleton. Let κ be an infinite regular cardinal and an upper bound on the cardinality of the set of morphisms in Sm_S, and hence on J. Every motivic symmetric spectrum is the filtered colimit of its β-bounded subobjects for any cardinal $\beta \geq \kappa$. Recall that E is β-bounded if the set $\bigcup_{m,n\geq 0, X\in \mathrm{Sm}_S} \mathrm{card}\,\big(E_n(X)\big)_m$ has cardinality at most β.

Example A.1: Every finitely presentable motivic symmetric spectrum is κ-bounded. To wit, if $X \in \mathrm{Sm}_S$ and K is a finite pointed simplicial set, then every finite colimit of κ-bounded motivic symmetric spectra of type $\mathrm{Fr}_n\, K \wedge X_+$ is κ-bounded. The notation Fr_n is standard for the left adjoint of the evaluation functor $E \longmapsto E_n$ for $n \geq 0$.

Lemma A.2: *Suppose E is β-bounded for $\beta \geq \kappa$ and F is finitely presentable. Then the set $\mathrm{Hom}_{\mathbf{MSS}_S}(F, E)$ has cardinality at most β.*

Proof. This holds by definition for $F = \mathrm{Fr}_n \Delta^m_+ \wedge X_+$ since Δ^m_+ is a finite simplicial set. The general case follows by passing to finite colimits. □

Example A.3: The image of a β-bounded motivic symmetric spectrum is β-bounded.

Proposition A.4: *The following statements hold for the fibrant replacement functor $\mathcal{R}$ and $\beta \geq \kappa$ a regular cardinal.*

1. *If $f\colon E \longrightarrow\!\!\!\!\!\rightarrow F$ is a monomorphism or cofibration, then so is $\mathcal{R}(f\colon E \longrightarrow\!\!\!\!\!\rightarrow F)$.*
2. *There is a natural isomorphism $\operatorname{colim}_{E' \subset E} \mathcal{R}(E') \xrightarrow{\cong} \mathcal{R}(E)$ where E' runs through the filtered category of β-bounded subspectra of E.*
3. *For monomorphisms $E \hookrightarrow G \hookleftarrow F$, $\mathcal{R}(E \cap F)$ coincides with the intersection $\mathcal{R}(E) \cap \mathcal{R}(F)$ in $\mathcal{R}(G)$.*
4. *If E is β-bounded, then so is $\mathcal{R}E$.*

Proof. It suffices to prove these claims for $\mathcal{R}^1$. For the first statement, observe that $\mathcal{R}^1(f)$ is obtained by taking pushouts in the diagram:

$$\begin{array}{ccccc}
\bigvee_{\coprod_{j\in J} \mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j, E)} & \mathrm{c}j \xleftarrow{\sim} & \bigvee_{\coprod_{j\in J} \mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j, E)} & \mathrm{d}j \longrightarrow & E \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
\bigvee_{\coprod_{j\in J} \mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j, F)} & \mathrm{c}j \xleftarrow{\sim} & \bigvee_{\coprod_{j\in J} \mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j, F)} & \mathrm{d}j \longrightarrow & F
\end{array} \tag{14}$$

Here, g is defined by $(\mathrm{d}j, \mathrm{d}j \xrightarrow{\alpha} E) \longmapsto (\mathrm{d}j, \mathrm{d}j \xrightarrow{\alpha} E \xrightarrow{f} F)$, and similarly for h. When f is a monomorphism (e.g. if f is a cofibration), then g and h are coproducts of cofibrations (recall that $\mathrm{d}j$ is cofibrant for every $j \in J$). Taking the pushout in the left hand square in (14) yields a map i, i.e. a coproduct of maps of the form j and $\mathrm{id}_{\mathrm{c}j}$. Thus $\mathcal{R}^1(f)$ is the composition of a cobase change of f and a cobase change of the acyclic cofibration i, hence a monomorphism and a cofibration if f is so. In view of the first part, the second claim follows easily by observing that any map $\alpha\colon \mathrm{d}j \longrightarrow\!\!\!\!\!\rightarrow E$ factors through some β-bounded subobject $E' \hookrightarrow E$ since $\mathrm{d}j$ is finitely presentable.

To prove the third claim, first note the inclusion i of $\mathcal{R}^1(E \cap F)$ into the pullback $\mathcal{R}^1(E) \cap \mathcal{R}^1(F)$ of $\mathcal{R}^1(E) \hookrightarrow \mathcal{R}^1(G) \hookleftarrow \mathcal{R}^1(F)$ is injective. Suppose (x, y) is an element in the codomain of i. Then either x is contained in E or in $\mathrm{c}j \smallsetminus \mathrm{d}j$ for some map $\alpha\colon \mathrm{d}j \longrightarrow E$, and likewise y is contained in F or $\mathrm{c}j' \smallsetminus \mathrm{d}j'$ for some $\alpha'\colon \mathrm{d}j' \longrightarrow F$. Since (x, y) is an element of the pullback, either $x = y \in F \cap G$ or $\mathrm{d}j \xrightarrow{\alpha} E \hookrightarrow G$ equals $\mathrm{d}j' \xrightarrow{\alpha'} F \hookrightarrow G$. In particular we get $j = j'$. Since the maps from E and F to G are monomorphisms, α and α' give rise to a map $\mathrm{d}j \longrightarrow E \cap F$, which implies that $(x, y) \in \mathcal{R}^1(E \cap F)$.

The last claim follows for $\mathcal{R}^1(E)$ by noting that $\mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j, E)$ is bounded by β. Since J has cardinality bounded by κ, the assumption implies

$$\bigvee_{\coprod_{j \in J} \mathrm{Hom}_{\mathbf{MSS}_S}(\mathrm{d}j, E)} \mathrm{c}j$$

is bounded by β, and hence the same holds for $\mathcal{R}^1(E)$. □

Corollary A.5: *Let F be a cofibrant finitely presentable motivic symmetric spectrum, and E be a β-bounded motivic symmetric spectrum, where $\beta \geq \kappa$ is a regular cardinal. Then $\mathrm{Hom}_{\mathbf{SH}(S)}(F, E)$ has cardinality at most β.*

Proof. The set $\mathrm{Hom}_{\mathbf{SH}(S)}(F, E)$ of homotopy classes of maps from F to $\mathcal{R}E$ is the quotient of a set of cardinality at most β since $\mathcal{R}E$ is β-bounded. □

A.2 The local model structure

Let L be a motivic symmetric spectrum and $(-)^c \longrightarrow \mathrm{Id}_{\mathbf{MSS}_S}$ the cofibrant replacement functor in the stable model structure on $\mathbf{MSS}_S$ obtained by applying the small object argument to the set of generating cofibrations

$$\mathrm{Fr}_m\big(X_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\big). \tag{15}$$

Note that if E is κ-bounded, then so is E^c.

Definition A.6: A map $f\colon E \longrightarrow F$ is an L-equivalence if $L \wedge f^c$ is a stable equivalence. It is an L-fibration if it has the right lifting property with respect to all maps that are both cofibrations and L-equivalences. A motivic symmetric spectrum E is L-acyclic if $* \longrightarrow E$ is an L-equivalence, and L-local if $\mathrm{Hom}_{\mathbf{SH}(S)}(F, E) = 0$ for every L-acyclic F.

Smashing with a cofibrant motivic symmetric spectrum preserves stable equivalences according to [10, 4.19]. Thus $L \wedge f^c$ is a stable equivalence if and only if $L^c \wedge f$ is a stable equivalence. In particular, every stable equivalence is an L-equivalence. It is immediate from Definition A.6 that the class of L-equivalences satisfy the two-out-of-three axiom.

Example A.7: If L is a monoid in $\mathbf{SH}(S)$, then every fibrant model in $\mathbf{MSS}_S$ for an L-module M in $\mathbf{SH}(S)$ is L-fibrant by an argument in [1]: If $E \rightarrowtail F$ is an L-acyclic cofibration, then to construct a lift in the diagram

$$\begin{array}{ccc} E & \longrightarrow & M \\ \downarrow & & \downarrow \\ F & \longrightarrow & * \end{array}$$

it suffices to prove that $f\colon \mathbf{sSet}_{\mathbf{MSS}_S}(F, M) \longrightarrow \mathbf{sSet}_{\mathbf{MSS}_S}(E, M)$ is surjective on zero-simplices. Since f is a Kan fibration, it suffices to show it is a weak equivalence. This follows provided every map of the form $G \wedge F/E \longrightarrow M$ is zero in $\mathbf{SH}(S)$, where G runs through a set of generators of $\mathbf{SH}(S)$. Every such map allows a factorization

$$G \wedge F/E \xrightarrow{\text{unit}} L \wedge G \wedge F/E \xrightarrow{L \wedge \alpha} L \wedge M \xrightarrow{\text{action}} M.$$

Now since $L \wedge F/E$ is trivial in $\mathbf{SH}(S)$ by assumption, the claim follows.

Lemma A.8: *Suppose G is a finitely presentable cofibrant motivic symmetric spectrum, $f\colon E \hookrightarrow F$ is an inclusion of motivic symmetric spectra and $i\colon W \hookrightarrow F$ is a subspectrum of cardinality* $\operatorname{card} W \leq \kappa$. *If $\alpha \in \operatorname{Hom}_{\mathbf{SH}(S)}(G, L \wedge W)$ is an element such that $\operatorname{Hom}_{\mathbf{SH}(S)}(G, L \wedge i)(\alpha)$ is contained in the image of $\operatorname{Hom}_{\mathbf{SH}(S)}(G, L \wedge f)$, there exists a factorization $W \xhookrightarrow{h} W' \hookrightarrow F$ of i such that W' is κ-bounded and $\operatorname{Hom}_{\mathbf{SH}(S)}(G, L \wedge h)(\alpha)$ is in the image of $\operatorname{Hom}_{\mathbf{SH}(S)}\big(G, L \wedge (E \cap W') \hookrightarrow L \wedge W'\big)$.*

Proof. The smash products in the statement of the Lemma are total left derived smash products. By [10, 4.19], they may be formed by smashing with a cofibrant replacement of L. Henceforth suppose that L is cofibrant. Let $a\colon G \longrightarrow \mathcal{R}(L \wedge W)$ in $\mathbf{MSS}_S$ be a representative of α. By assumption there exists a homotopy $H\colon G \wedge \Delta^1_+ \longrightarrow \mathcal{R}(L \wedge F)$ between $G \xrightarrow{a} \mathcal{R}(L \wedge W) \hookrightarrow \mathcal{R}(L \wedge F)$ and $G \xrightarrow{b} \mathcal{R}(L \wedge E) \hookrightarrow \mathcal{R}(L \wedge F)$ for some b. Smashing with L preserves colimits, so Part 2 of Proposition A.4 shows $\mathcal{R}(L \wedge F)$ is the filtered colimit (union) of objects $\mathcal{R}(L \wedge W')$, where W' is a κ-bounded

subspectrum of F containing W. Since $G \wedge \Delta^1_+$ is finitely presentable, the homotopy H factors through $\mathcal{R}(L \wedge W')$. Note that one end of the homotopy $G \wedge \Delta^1_+ \longrightarrow \mathcal{R}(L \wedge W')$ is the composite $G \xrightarrow{a} \mathcal{R}(L \wedge W) \hookrightarrow \mathcal{R}(L \wedge W')$, while the other end has a factorization $G \xrightarrow{c} \mathcal{R}(L \wedge E) \cap \mathcal{R}(L \wedge W') \hookrightarrow \mathcal{R}(L \wedge W')$. Part 3 of Proposition A.4 and the fact that smashing with a cofibrant spectrum commute with intersections, see Lemma A.9 below, imply

$$\mathcal{R}(L \wedge E) \cap \mathcal{R}(L \wedge W') = \mathcal{R}\big((L \wedge E) \cap (L \wedge W')\big) = \mathcal{R}\big(L \wedge (E \cap W')\big).$$

This shows that c represents the desired element. □

Lemma A.9: *Suppose E and F are subspectra of a motivic symmetric spectrum G. If L is a cofibrant motivic symmetric spectrum, then $L \wedge (E \cap F)$ coincides with the intersection of $L \wedge E$ and $L \wedge F$ in $L \wedge G$.*

Proof. Recall from [10, 4.19] that smashing with L preserves monomorphisms since L is cofibrant. Thus $L \wedge (E \cap F)$ injects into $(L \wedge E) \cap (L \wedge F)$. To prove surjectivity, let

$$\begin{array}{ccccc} B_1 & \leftarrowtail & A_1 & \hookrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ B_0 & \leftarrowtail & A_0 & \hookrightarrow & C_0 \\ \uparrow & & \uparrow & & \uparrow \\ B_2 & \leftarrowtail & A_2 & \hookrightarrow & C_2 \end{array} \tag{16}$$

be a diagram of sets such that $A_0 \cup_{A_1} C_1 \longrightarrow C_0$ and $A_0 \cup_{A_2} C_2 \longrightarrow C_0$ are injective. Then the pullback (intersection) of $B_1 \cup_{A_1} C_1 \hookrightarrow B_0 \cup_{A_0} C_0 \hookleftarrow B_2 \cup_{A_2} C_2$ coincides with the pushout of $B_1 \cap B_2 \leftarrowtail A_1 \cap A_2 \hookrightarrow C_1 \cap C_2$. Since pushouts and intersections in $\mathbf{MSS}_S$ are ultimately computed in the category of sets, the same statement holds for motivic symmetric spectra. Suppose $L = \mathrm{Fr}_n A$, where A is a motivic space. Then

$$\big(L \wedge E\big)_{n+m} = \begin{cases} \Sigma^+_{n+m} \wedge_{\{1\} \times \Sigma_m} A \wedge E_m & m \geq 0 \\ * & m < 0 \end{cases}$$

for every motivic symmetric spectrum E. Now since smashing with every motivic space preserves intersections, as one may deduce from (16), it follows that smashing with $\mathrm{Fr}_n A$ commutes with intersections. And by attaching cells and contemplating (16), the result follows for arbitrary cofibrant motivic symmetric spectra. □

Lemma A.8 can be iterated for every collection of elements α with finitely presentable cofibrant domain which is bounded by κ, using that κ is regular.

Lemma A.10: *Suppose $f\colon E \hookrightarrow F$ is an inclusion of motivic symmetric spectra which is not an isomorphism and an L-equivalence. Then there exists an injection $i\colon W \hookrightarrow F$ with the following properties.*

1. *$E \cap W \hookrightarrow W$ is not an isomorphism.*
2. *$E \cap W \hookrightarrow W$ is an L-equivalence.*
3. *The cardinality of W is bounded by κ.*

Proof. The category $\mathbf{SH}(S)$ is weakly generated by shifted suspension spectra $\mathrm{Fr}_n X_+$, where $n \geq 0$ and $X \in \mathrm{Sm}_S$. In other terms, a motivic symmetric spectrum E is trivial in $\mathbf{SH}(S)$ if and only if $\mathrm{Hom}_{\mathbf{SH}(S)}(\mathrm{Fr}_n S^m \wedge X_+, E)$ is trivial for all $m, n \geq 0$, $X \in \mathrm{Sm}_S$. We note that the weak generators are small. Now choose an inclusion $z\colon W_0 \hookrightarrow F$ of motivic symmetric spectra such that Parts 1 and 3 hold true. For example, one can choose an m-simplex in $F_n(X)$ which is not in the image of f and consider the image of $\mathrm{Fr}_n \Delta^m_+ \wedge X_+ \longrightarrow F$. For every element $\alpha \in \mathrm{Hom}_{\mathbf{SH}(S)}(G, L \wedge W_0)$ there exists a κ-bounded subspectrum $W_\alpha \hookrightarrow F$ containing W_0 such that the image of α in $L \wedge W_\alpha$ is contained in the image of $L \wedge (E \cap W_\alpha) \longrightarrow L \wedge W_\alpha$. Then $W_1 \equiv \bigcup W_\alpha$ is κ-bounded and iterating this procedure for the cardinality of the natural numbers produces W. □

Corollary A.11: *Suppose $p\colon X \longrightarrow Y$ has the right lifting property with respect to all L-acyclic monomorphisms with κ-bounded codomain. Then p has the right lifting property with respect to all L-acyclic monomorphisms.*

Proof. Follows from Lemma A.10 using left properness and a Zorn's lemma argument as in the proof of [7, 1.1]. □

Corollary A.12: *Every map in $\mathbf{MSS}_S$ acquires a functorial factorization into an L-acyclic cofibration composed with an L-fibration.*

Proof. First, a small object argument shows that every map factorizes into an L-acyclic monomorphism i composed with a map p having the right lifting property with respect to κ-bounded L-acyclic monomorphisms. To show i is an L-acyclic monomorphism, we rely on the facts that smashing with a cofibrant motivic symmetric spectrum preserves

monomorphisms and that acyclic monomorphisms are closed under cobase change, see [10, 4.15, 4.19]. Corollary A.11 shows that p is an L-fibration. Second, we may factorize the L-acyclic monomorphism into a cofibration composed with an acyclic fibration and note that acyclic fibrations are L-fibrations. □

Theorem A.13: *The classes of cofibrations, L-equivalences and L-fibrations define a left proper cofibrantly generated monoidal and simplicial model structure on* $\mathbf{MSS}_S$.

Proof. The model category axioms **CM1**-**CM3** hold trivially and Corollary A.12, which is the interesting part of the factorization axiom **CM5**, implies the lifting axiom **CM4** by a standard argument. Smashing a stable acyclic cofibration with a motivic symmetric spectrum yields a stable equivalence and the stable model structure is monoidal, so the L-local model structure is monoidal; in particular, the model structure is simplicial. It is also standard to deduce left properness.

To construct generating L-acyclic cofibrations, take the union of the set of generating acyclic cofibrations in the stable model structure and some set J_L of representatives of isomorphism classes of L-acyclic cofibrations with κ-bounded codomains. Suppose that $p\colon X \longrightarrow\!\!\!\!\!\rightarrow Y$ is a stable fibration having the right lifting property with respect to J_L. Given a lifting problem

$$\begin{array}{ccc} E & \longrightarrow & X \\ {\scriptstyle L\sim}\downarrow & & \downarrow{\scriptstyle p} \\ F & \longrightarrow & Y \end{array} \tag{17}$$

choose a κ-bounded subspectrum $W \hookrightarrow F$ such that $i\colon E \cap W \hookrightarrow W$ is L-acyclic. Factor i as a cofibration $j\colon E \cap W \rightarrowtail G$ followed by an acyclic fibration $G \xrightarrow{\sim}\!\!\!\!\!\rightarrow W$ via the small object argument applied to the set (15). Note that j is an L-acyclic cofibration with a κ-bounded codomain. Hence, by the assumption on p, there exists a lift in the diagram:

$$\begin{array}{ccc} E \cap W & \longrightarrow & X \\ {\scriptstyle j}\downarrow & {\scriptstyle L\sim} & \downarrow{\scriptstyle p} \\ G & \longrightarrow & Y \end{array}$$

Thus there exists a lift $G \cup_{E\cap W} E \longrightarrow X$. Since stable equivalences are preserved under cobase change along monomorphisms, the induced map $G \cup_{E\cap W} E \longrightarrow F$ is a stable equivalence. It factors as a cofibration followed by an acyclic fibration $q\colon H \xrightarrow{\sim}\!\!\!\!\!\rightarrow F$.

Using that p is a stable fibration, there exists a lift $H \longrightarrow\!\!\!\!\succ X$, and since $E \rightarrowtail\!\!\!\!\succ F$ is a cofibration, there exists a lift in the diagram:

$$\begin{array}{ccc} E & \longrightarrow\!\!\!\!\succ & H \\ {\scriptstyle L}\sim\big\downarrow & & \sim\big\downarrow{\scriptstyle q} \\ F & \xrightarrow{\mathrm{id}}\!\!\!\!\succ & F \end{array}$$

The lifting problem (17) can be resolved by combining the liftings constructed above. □

Remark A.14: The proof of Theorem A.13 shows the L-local model structure coincides with the left Bousfield localization of $\mathbf{MSS}_S$ with respect to the set of representatives of isomorphism classes of L-acyclic cofibrations with κ-bounded codomains.

References

[1] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[2] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[3] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[4] Bjørn Ian Dundas, Oliver Röndigs, and Paul Arne Østvær. Motivic functors. *Doc. Math.*, 8:489–525 (electronic), 2003.

[5] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:No. 4, 107–131 (electronic), 2003.

[6] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 1998.

[7] P. G. Goerss and J. F. Jardine. Localization theories for simplicial presheaves. *Canad. J. Math.*, 50(5):1048–1089, 1998.

[8] Christian Hasemeyer and Jens Hornbostel. Motives and etale motives with finite coefficients. *K-Theory*, 34(3):195–207, 2005.

[9] Po Hu. On the Picard group of the stable $\mathbb{A}^1$-homotopy category. *Topology*, 44(3):609–640, 2005.

[10] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[11] Fabien Morel. On the motivic π_0 of the sphere spectrum. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 219–260. Kluwer Acad. Publ., Dordrecht, 2004.

[12] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[13] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[14] Ivan Panin and Serge Yagunov. Rigidity for orientable functors. *J. Pure Appl. Algebra*, 172(1):49–77, 2002.

[15] Oliver Röndigs and Paul Arne Østvær. Motives and modules over motivic cohomology. *C. R. Math. Acad. Sci. Paris*, 342(10):751–754, 2006.

[16] A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math.*, 73(2):241–245, 1983.

[17] Vladimir Voevodsky. Cancellation theorem. Preprint.

[18] Vladimir Voevodsky. Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (98):59–104, 2003.

[19] Serge Yagunov. Rigidity. II. Non-orientable case. *Doc. Math.*, 9:29–40 (electronic), 2004.

Institute of Mathematics, University of Osnabrück, Osnabrück, Germany.
e-mail: oroendig@math.uni-osnabrueck.de

Department of Mathematics, University of Oslo, Oslo, Norway.
e-mail: paularne@math.uio.no