Lecture Notes

Selected Topics in M214

G. Berge

October 24, 2003

Lectures autumn 2002.

This comes in addition to the textbook by Alan W. Bush.

Preface

These Lecture Notes comes in addition to the textbook by Alan W. Bussh: Perturbation Methods for Engeneers and Scientists. CRC Press.

Department of Matematics, University of Bergen September 2002
Last correction 17.10.2003
Gerhard Berge

Contents

1 Ordering relations. Asymptotic series **1**
1.1 Ordering-symbols . . . 1
1.2 Examples . . . 2
1.2.1 Combinations of ordering relations . . . 2
1.3 Asymptotic sequences, expansion, and series . . . 2
1.3.1 Dealing with asymptotic series . . . 3
1.3.2 Coefficients in an asymptotic expansion, series . . . 3
1.3.3 Asymptotic expansion of an integral . . . 4

2 The averaging method **6**
2.1 Introductory remarks . . . 6
2.2 The averaging method . . . 7
2.3 Some examples . . . 9

3 Dynamic systems **14**
3.1 Introduction . . . 14
3.2 General systems, phase space . . . 14
3.2.1 Existence and uniqueness . . . 15
3.3 A fluid picture . . . 17
3.4 Conservative, dissipative systems . . . 19
3.4.1 Hamiltonian systems . . . 19
3.5 Equilibrium and stability . . . 21
3.5.1 Stability of an equilibrium . . . 21
3.5.2 Classifying equilibria in two dimensions . . . 22
3.6 Orbital stability . . . 27
3.7 Liapunov stability . . . 28
3.7.1 Liapunov-exponents . . . 29
3.7.2 A special class of linear systems (I) . . . 30
3.7.3 Structure of a n-dimensional linear system . . . 30
3.7.4 A special class of linear systems II . . . 33
3.8 Linear systems, periodic coefficients . . . 35
3.8.1 Stability of a periodic system . . . 38
3.9 Liapunov's direct method . . . 41
3.9.1 Positive definite functions, properties . . . 42
3.9.2 Liapunov's stability conditions . . . 42
3.10 Stability of the linear approximation . . . 47
3.11 Attractors . . . 51
3.12 Poincaré-cut . . . 54

4 Universal non linear processes **56**
4.1 Period-doubling Cascade to Chaos . . . 56
4.1.1 Functional iterations . . . 58
4.1.2 Fixed-point behavior under functional iteration . . . 59
4.2 Universal limit for high iteration . . . 68
4.3 Bifurcations . . . 70
4.3.1 Bifork bifurcation . . . 70
4.3.2 Hopf-bifurcation . . . 70
4.4 Influence from small differences in initial conditions . . . 71

4.5 "Strange" attractors and fractals 74
4.6 Dimension 74
4.6.1 Capacity-dimension 75
4.7 Henon attractor 76
4.8 Three Period Evolution 77
4.9 Deterministic chaos 79
4.10 Practical use 81

A Existence and uniqueness of solution of first order ordinary differential equations (ODE) 84
A.1 Successive approximations or Picard iterations 84
A.2 Lipschitz-condition 85
A.3 Existence and uniqueness theorem 86

B Superstable trajectories 89

C The universal constant α 92

1 Ordering relations. Asymptotic series

In the following ϕ and ψ are functions of x and ϵ where x can be a vector or scaler and ϵ is a scaler.

1.1 Ordering-symbols

In the following let $\epsilon \in I$ and $x \in D$:

Definition I: Big $\mathcal{O}$

We have $\phi \stackrel{def}{=} \mathcal{O}(\psi)$ if for a fixed value for x there exist $k(x) < \infty$ and an ϵ-neighborhood of ϵ_0, N_ϵ, such that

$$\epsilon \in I \cap N_\epsilon \quad \Rightarrow \quad |\phi| \leq k(x)|\psi| \,. \tag{1}$$

If $\psi \neq 0$ in the actual region, this is equivalent to that the limit

$$\lim_{\epsilon \to \epsilon_0} \frac{\phi}{\psi} \tag{2}$$

exists.

Definition II: Little o

We have $\phi \stackrel{def}{=} o(\psi)$ if there for a fixed value of x and an arbitrary $\delta(x) > 0$ exist an ϵ-neighborhood of ϵ_0, N_δ, such that

$$\epsilon \in I \cap N_\delta \quad \Rightarrow \quad |\phi| \leq \delta(x)|\psi| \,. \tag{3}$$

If $\psi \neq 0$ in the actual region, then this is equivalent to the limit

$$\lim_{\epsilon \to \epsilon_0} \frac{\phi}{\psi} = 0 \,. \tag{4}$$

Definition III: Uniformity

If the quantities $k(x)$ and $\delta(x)$ and the regions N_ϵ and N_δ above can be chosen such that these relations are valid for all $x \in D$, then we say that the ordering relations are uniformly valid in D.

We shall study several examples where this is not true, and where the reason can be attributed to nonuniform behavior.

1.2 Examples

$$
\begin{aligned}
\sin\epsilon &= \mathcal{O}(\epsilon)\,, \quad \epsilon \to 0\,,\\
\cos\epsilon &= \mathcal{O}(1)\,, \quad \epsilon \to 0\,,\\
\sinh\epsilon &= \mathcal{O}(\epsilon)\,, \quad \epsilon \to 0\,,\\
\sinh\frac{1}{\epsilon} &= \mathcal{O}(e^{\frac{1}{\epsilon}})\,, \quad \epsilon \to 0\,,\\
\frac{\epsilon^{\frac{3}{2}}}{\sin\epsilon} &= \mathcal{O}(\epsilon^{\frac{1}{2}})\,, \quad \epsilon \to 0\,,\\
\sin\epsilon &= o(1)\,, \quad \epsilon \to 0\,,\\
\sin\epsilon &= o(\epsilon^{\frac{1}{2}})\,, \quad \epsilon \to 0\,,\\
e^{\frac{2}{\epsilon}} &= o(e^{e^{\frac{1}{\epsilon}}})\,, \quad \epsilon \to 0\,.
\end{aligned}
$$

1.2.1 Combinations of ordering relations

The following relations can easily be proved to be true by using the definitions of the ordering symbols.

$$
\begin{aligned}
\mathcal{O}(\mathcal{O}(\phi)) &= \mathcal{O}(\phi)\,,\\
\mathcal{O}(o(\phi)) &= o(\mathcal{O}(\phi)) = o(o(\phi)) = o(\phi)\,,\\
\mathcal{O}(\phi)\mathcal{O}(\psi) &= \mathcal{O}(\phi\psi)\,,\\
\mathcal{O}(\phi)o(\psi) &= o(\phi)o(\psi) = o(\phi\psi)\,,\\
\mathcal{O}(\phi) + \mathcal{O}(\phi) &= \mathcal{O}(\phi) + o(\phi) = \mathcal{O}(\phi)\,,\\
o(\phi) + o(\phi) &= o(\phi)\,.
\end{aligned}
$$

1.3 Asymptotic sequences, expansion, and series

Let the sequence $\{\phi_n(\epsilon)\}, \quad n = 1, 2, \ldots$ where the $\phi_n(\epsilon)$ are given as functions of ϵ .

Definition IV: Asymptotic sequence

The sequence $\{\phi_n(\epsilon)\}$ is an asymptotic sequence if for all n, we have $\phi_{n+1}(\epsilon) = o(\phi_n(\epsilon)$ when $\epsilon \to \epsilon_0$.

REMARK: If δ and N_δ, related to the ordering relation "o" above in correspondence with the definitions, can be made independent of n, then we say that the asymptotic sequence is uniform in n. Correspondingly we say that the sequence is uniform in x if the corresponding requirements are valid with respect to x.

Asymptotic sequences; two examples:

$$
\begin{aligned}
\phi_n(\epsilon) &= (\epsilon - \epsilon_0)^n \quad \text{når} \quad \epsilon \to \epsilon_0\\
\phi_n(\epsilon) &= e^{\epsilon}\epsilon^{-\lambda_n} \quad \text{når} \quad \epsilon \to \infty \quad \& \ \lambda_{n+1} > \lambda_n
\end{aligned}
$$

1.3.1 Dealing with asymptotic series

We can multiply two asymptotic series and this way construct a new asymptotic series. We can integrate an asymptotic series term by term. As a general rule we can not take the derivative of an asymptotic series term by term. Make a comparison to the corresponding problem for Fourier series.

Definition V: Asymptotic expansion, series

A sum of N terms

$$\sum_{n=1}^{N} a_n(x)\phi_n(\epsilon),$$

makes an asymptotic expansion (series) of the function $f(x,\epsilon)$ *up to N terms if*

$$f(x,\epsilon) - \sum_{n=1}^{N} a_n(x)\phi_n(\epsilon) = o(\phi_N(\epsilon)) \quad \text{when} \quad \epsilon \to \epsilon_0\,, \tag{5}$$

and this relation is valid stepwise $n = 1,\, 2,\, 3 \ldots N$.

For the limit $N \to \infty$ we shall use the notation

$$f(x,\epsilon) \sim \sum_{n=1}^{\infty} a_n(x)\phi_n(\epsilon)\,. \tag{6}$$

The symbol "$\sim$" we read as "asymptotic equal". An asymptotic expansion is uniformly valid in a region D in x, if these relations are valid uniformly for $x \in D$.

1.3.2 Coefficients in an asymptotic expansion, series

The coefficients in an asymptotic expansion (series) can be found by the following limiting processes, and the result is unique

$$a_1(x) = \lim_{\epsilon\to\epsilon_0} \frac{f(x,\epsilon)}{\phi_1(\epsilon)} \tag{7}$$

$$a_2(x) = \lim_{\epsilon\to\epsilon_0} \frac{f(x,\epsilon) - a_1(x)\phi_1(\epsilon)}{\phi_2(\epsilon)} \tag{8}$$

$$\vdots$$

$$a_N(x) = \lim_{\epsilon\to\epsilon_0} \frac{f(x,\epsilon) - \sum_{n=1}^{N-1} a_n(x)\phi_n(\epsilon)}{\phi_N(\epsilon)} \tag{9}$$

Example:

$$(x+\epsilon)^{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \prod_{k=1}^{n} |2k-3| \frac{\epsilon^{n-1}}{x^{(2n-1)/2}} \quad \epsilon \to 0\,,$$

with respect to the sequence $\phi_n = \epsilon^{n-1}$. Notice that this also represent a Taylor expansion around $\epsilon = 0$. This particular series is convergent for $|x| > \epsilon$, and it is not uniform in regions where $x = 0$ is a limit point. Check this result!

1.3.3 Asymptotic expansion of an integral

If $f(x,\epsilon)$ is given as an integral, then we have a different kind of problems which require different techniques. For example we have partial integration as a very useful technique. We take a look at the following example (the complementary error function):

$$\mathrm{erfc}(\epsilon) \stackrel{def}{=} 1 - \frac{2}{\sqrt{\pi}} \int_{\epsilon}^{\infty} e^{-t^2} dt \,. \tag{10}$$

We look for an asymptotic expansion of this integral for large ϵ. First let $t^2 = \tau$ and we find

$$\mathrm{erfc}(\epsilon) = 1 - \frac{1}{\sqrt{\pi}} \int_{\epsilon^2}^{\infty} \frac{e^{-\tau}}{\sqrt{\tau}} d\tau \,. \tag{11}$$

After a partial integration we obtain

$$\mathrm{erfc}(\epsilon) = 1 - \frac{1}{\sqrt{\pi}} \left[\frac{e^{-\epsilon^2}}{\epsilon} - \frac{1}{2} \int_{\epsilon^2}^{\infty} e^{-\tau} \tau^{-\frac{3}{2}} d\tau \right] . \tag{12}$$

This result suggests that it might be possible to generate this asymptotic sequence in increasing powers in $\frac{1}{\epsilon}$ term by term by partial integration. Such a series will be of value in order to evaluate $\mathrm{erf}(\epsilon)$ for large ϵ.

We continue by a definition

$$F_n(\epsilon) \stackrel{def}{=} \int_{\epsilon}^{\infty} e^{-\tau} \tau^{-\frac{2n+1}{2}} d\tau \,, \quad n = 0,\, 1,\, 2,\, \ldots \,. \tag{13}$$

From partial integration we now obtain

$$F_n(\epsilon) = \frac{e^{-\epsilon^2}}{\epsilon^{2n+1}} - \frac{2n+1}{2} F_{n+1}(\epsilon) \,, \quad n = 0,\, 1,\, 2,\, \ldots \,. \tag{14}$$

This recurrency relation contains the result

$$\begin{aligned} F_0(\epsilon) &= e^{-\epsilon^2} \left[\frac{1}{\epsilon} - \frac{1}{2\epsilon^3} + \frac{1 \cdot 3}{2^2 \epsilon^5} + \cdots + \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^{n-1} \epsilon^{2n-1}} \right] \\ &\quad + (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)}{2^n} F_n(\epsilon) \,, \quad n = 0,\, 1,\, 2,\, \ldots \end{aligned} \tag{15}$$

Check this result!

One can use the definition of an asymptotic series to show that the representation given by Eq. (15) is an asymptotic series. Do this as an exercise! (Otherwise look in Bush page 34 for an alternative computation.)

We notice that computing asymptotic series quickly leads to extensive computations. It is also the case that one rarely finds ready made recipes for such work. This part of mathematics is therfore often more an "art" than a "craftsmanship".

Termination rule It is an important problem to know where or when to terminate an asymptotic series. Such a series is normally not convergent, therefore after a while it becomes meaningless to take into account more terms. A useful practical rule is the following:

One keeps terms up to the term (not including) which is least in magnitude. The first term omitted is normally a measure for the error, and we want to make the error as small as possible, therfore this rule.

2 The averaging method

The theory presented here is according to a lecture note from 1989.

2.1 Introductory remarks

We start by looking at a problem where a certain class of equations can be rewritten to a standard form applicable to the *averaging method*. The starting point is the equation

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0\,, \tag{16}$$

We change over to the phase plane

$$\begin{aligned} \dot{x} &= y\,, & (17)\\ \dot{y} &= -x - \epsilon h(x, y)\,, & (18) \end{aligned}$$

and introduce polar coordinates in the phase plane

$$\begin{aligned} x = r\cos\theta\,, \quad \dot{x} = \dot{r}\cos\theta - r\sin\theta\ \dot{\theta}\,, & \qquad (19)\\ y = r\sin\theta\,, \quad \dot{y} = \dot{r}\sin\theta + r\cos\theta\ \dot{\theta}\,. & \qquad (20) \end{aligned}$$

These equations can also be written as

$$\begin{aligned} \dot{r} &= -\epsilon h(r\cos\theta, r\sin\theta)\sin\theta\,, & (21)\\ \dot{\theta} &= -1 - \frac{\epsilon}{r} h(r\cos\theta, r\sin\theta)\cos\theta\,, & (22) \end{aligned}$$

or

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{\epsilon h(r\cos\theta, r\sin\theta)\sin\theta}{1 + \frac{\epsilon}{r} h(r\cos\theta, r\sin\theta)\cos\theta}\,. \tag{23}$$

We introduce new variables

$$\alpha = \theta + t\,, \quad \dot{\alpha} = \dot{\theta} + 1\,, \tag{24}$$

and can then write

$$\begin{aligned} \dot{r} &= -\epsilon h\{r\cos(\alpha - t), r\sin(\alpha - t)\}\sin(\alpha - t)\,, & (25)\\ \dot{\alpha} &= -\frac{\epsilon h}{r}\{r\cos(\alpha - t), r\sin(\alpha - t)\}\cos(\alpha - t)\,. & (26) \end{aligned}$$

Both Eq. (23) and the system Eqs.(25) and (26) are of the type

$$\dot{\mathbf{x}} = \epsilon \mathbf{F}(\mathbf{x}, t)\,, \tag{27}$$

where $\mathbf{x}$ is a vector

$$\mathbf{x} = \left\{ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\}, \qquad \dot{\mathbf{x}} = \left\{ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{array} \right\}. \tag{28}$$

In the case of Eq.(23) this vector has only one component (that is it is a scalar), in the case Eqs. (25) (26) this vector has two components (r, α). Also notice that by the transformation from Eqs. (17), (18) to Eqs. (25), (26) we move from an autonomous to a non autonomous system. Especially we notice that Eq. (27) is a non autonomous system.

We shall study equations of the type Eq. (27) by the aid of an asymptotic method developed by Boguliubov and Krylov [1].

2.2 The averaging method

The averaging method can be used to find asymptotic solutions to systems having two distinct timescales, one short and one long. For example systems that are oscillatory where for example the amplitude varies slowly on the timescale corresponding to one period of the oscillation. Eq. (23) has a form which shows that r varies slowly as a function of θ, whereas h varies "fast" as a function of θ over the interval $(0, 2\pi)$. It may therefore be meaningful to look for a solution $r(\theta)$, which from the outset represent some sort of mean value (over θ) of the exact solution.

If we look at the system Eqs. (25), (26), the same is true here. Both r and α varies slowly as function of t, whereas the components in $\mathbf{F}$,

$$-h\{r\cos(\alpha - t), r\sin(\alpha - t)\}\sin(\alpha - t)\,, \tag{29}$$

and

$$-h\{r\cos(\alpha - t), r\sin(\alpha - t)\}\cos(\alpha - t)\,, \tag{30}$$

varies "fast" with respect to t. It is therfore natural to start looking for a representation of the solution that as a start behave as an average or mean value of r and α as functions of t. In order to achieve this according to the method of Boguliubov and Krylov we solve Eq. (27) as follows:

As a start we find a Fourier expansion of the periodic function $\mathbf{F}(\mathbf{x},t)$ in the variable t, corresponding to the "fast" variations (notice that this is a formal Fourier expansion were $\mathbf{x}$ and t is to be regarded as independent variables).

$$\mathbf{F}(\mathbf{x},t) = \mathbf{F}_0(\mathbf{x}) + \sum_{n\neq 0} \mathbf{F}_n(\mathbf{x}) e^{int}\,. \tag{31}$$

Furthermore we define

$$\tilde{\mathbf{F}}(\mathbf{x},t) = \sum_{n\neq 0} \frac{1}{in} \mathbf{F}_n(\mathbf{x}) e^{int}\,. \tag{32}$$

Notice that this implies

$$\frac{\partial \tilde{\mathbf{F}}(\mathbf{x},t)}{\partial t} = \sum_{n\neq 0} \mathbf{F}_n(\mathbf{x}) e^{int} = \mathbf{F}(\mathbf{x},t) - \mathbf{F}_0(\mathbf{x})\,. \tag{33}$$

We introduce the symbol $\underset{t}{M}\{\ \}$ as the following operation

$$\underset{t}{M}\{\mathbf{F}(\mathbf{x},t)\} \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}(\mathbf{x},t)dt = \mathbf{F}_0(\mathbf{x})\,, \tag{34}$$

notice that $\mathbf{x}$ is considered to be constant under this integration (for $\underset{t}{M}\{\ \}$ read the average over t of $\{\ \}$).

We define at this point an implicit transformation $\boldsymbol{\xi}(t)$ by the equation

$$\mathbf{x}(t) = \boldsymbol{\xi}(t) + \epsilon\tilde{\mathbf{F}}(\boldsymbol{\xi},t)\,, \tag{35}$$

and find

$$\frac{d\mathbf{x}(t)}{dt} = \frac{d\boldsymbol{\xi}}{dt} + \epsilon\frac{d\boldsymbol{\xi}}{dt}\cdot\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial\boldsymbol{\xi}} + \epsilon\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial t}\,. \tag{36}$$

Substituted in Eq.(27) this gives

$$\begin{aligned}\frac{d\boldsymbol{\xi}}{dt}\cdot\{\mathbf{I}+\epsilon\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial\boldsymbol{\xi}}\} &= -\epsilon\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial t} + \epsilon\mathbf{F}\{\boldsymbol{\xi}+\epsilon\tilde{\mathbf{F}}(\boldsymbol{\xi},t)\}\\ &= \epsilon\{-\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial t} + \mathbf{F}(\boldsymbol{\xi},t)\} + \mathcal{O}(\epsilon^2)\,,\end{aligned}$$

here $\mathbf{I}$ is the unit matrix and we find

$$\begin{aligned}\frac{d\boldsymbol{\xi}}{dt} &= \epsilon\{-\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial t} + \mathbf{F}(\boldsymbol{\xi},t)\}\{\mathbf{I}+\epsilon\frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial t}\}^{-1} + \mathcal{O}(\epsilon^2)\\ &= \epsilon\{\mathbf{F}(\boldsymbol{\xi},t) - \frac{\partial\tilde{\mathbf{F}}(\boldsymbol{\xi},t)}{\partial t}\} + \mathcal{O}(\epsilon^2)\\ &= \epsilon\mathbf{F}_0(\boldsymbol{\xi}) + \mathcal{O}(\epsilon^2).\end{aligned}$$

In the last step Eq. (33) is used and this step shows why it is convenient to define the transformation as we did in Eq. (35). The first approximation we find by solving the equation is

$$\frac{d\boldsymbol{\xi}}{dt} = \epsilon\mathbf{F}_0(\boldsymbol{\xi}) = \epsilon\ \underset{t}{M}\{\mathbf{F}(\boldsymbol{\xi},t)\}\,, \tag{37}$$

thus the equation we obtain by starting from the exact equation and replace the lefthand side by the function obtained by averaging over t. This is the reason for the name *the averaging method*. By assumption $\mathbf{F}(\boldsymbol{\xi},t)$ is a 2π- periodic function of t. The improved first approximation, which is correct to order ϵ in the asymptotic sense, can be found by taking this solution for $\boldsymbol{\xi}$ and substitute in

$$\mathbf{x}(t) = \boldsymbol{\xi}(t) + \epsilon\tilde{\mathbf{F}}(\boldsymbol{\xi},t)\,,$$

where $\mathbf{x}(t)$ now is an approximation correct to order ϵ.

One can continue using this method and define a transformation which gives a solution correct to order ϵ^2. In principle one can obtain solutions correct to any order in ϵ. However, in actual problems it quickly becomes so complicated that for practical reasons it becomes very difficult to continue. The resulting

differential equations is not solvable, or the algebra is not manageable. Mathematically this method is well funded, but we are not pursuing this here. For most practical problems one is satisfied by obtaining the solution correct to order ϵ.

2.3 Some examples

Example 2.1 (Van der Pol's equation) *Find an approximate solution to Van der Pol's equation*

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{38}$$

for small ϵ.

We change to polar coordinates and find

$$\begin{aligned} \dot{r} &= -\epsilon h(r\cos\theta, r\sin\theta)\sin\theta\,, & (39)\\ \dot{\theta} &= -1 - \frac{\epsilon}{r}h(r\cos\theta, r\sin\theta)\cos\theta\,, & (40) \end{aligned}$$

or

$$\frac{dr}{d\theta} = \frac{\epsilon h(r\cos\theta, r\sin\theta)\sin\theta}{1 + \frac{\epsilon}{r}h(r\cos\theta, r\sin\theta)\cos\theta}\,. \tag{41}$$

First we look at the system (39) and (40) and put $\theta = \alpha - t$ *and* $\dot{\theta} = \dot{\alpha} - 1$. *We find*

$$\begin{aligned} \dot{r} &= -\epsilon h(r\cos\theta, r\sin\theta)\sin\theta = \dot{x}_1\,,\\ \dot{\alpha} &= -\epsilon h(r\cos\theta, r\sin\theta)\cos\theta = \dot{x}_2\,, \end{aligned}$$

were $x_1 = r$ *and* $x_2 = \alpha$, *furthermore*

$$h(r\cos\theta, r\sin\theta) = (r^2\cos^2\theta - 1)r\sin\theta = (r^2\cos^2(\alpha - t) - 1)r\sin(\alpha - t)\,, \tag{42}$$

$$\begin{aligned} F_1 &= -h\sin\theta = -[r^2\cos^2(\alpha - t) - 1]r\sin^2(\alpha - t)\,,\\ F_2 &= -h\cos\theta = -[r^2\cos^2(\alpha - t) - 1]\sin(\alpha - t)\cos(\alpha - t)\,, \end{aligned}$$

$$\underset{t}{M}\{F_1\} = -\frac{1}{2\pi}\int_0^{2\pi}[r^2\cos^2(\alpha - t) - 1]r\sin^2(\alpha - t)dt\,,$$

notice that α *and* r *are considered to be constant in this integral. A new variable* $\alpha - t = u$ *gives*

$$\begin{aligned} \underset{t}{M}\{F_1\} &= \frac{1}{2\pi}\int_{\alpha}^{\alpha - 2\pi}[r^2\cos^2 u - 1]r\sin^2 u du\\ &= \frac{1}{2}(r - \frac{1}{4}r^3)\,,\\ \underset{t}{M}\{F_2\} &= 0\,, \end{aligned}$$

$$\begin{aligned}\frac{d\xi_1}{dt} &= \frac{\epsilon}{2}\xi_1(1-\frac{1}{4}\xi_1^2)\,,\\ \frac{d\xi_2}{dt} &= 0\,.\end{aligned}$$

Here ξ_1 is a first approximation to r and ξ_2 is a first approximation to α. We find

$$\xi_1 = \frac{2}{[1-(1-\frac{4}{a_0^2})e^{-\epsilon t}]^{\frac{1}{2}}} \tag{43}$$

$$\xi_2 = konst. = \theta_0 \tag{44}$$

where $\xi_1(0) = a_0$. Thus as a first approximation we have:

$$r = \xi_1\,, \tag{45}$$

$$\alpha = \theta_0 \quad \text{eller} \quad \theta = \theta_1 = \theta_0 - t\,. \tag{46}$$

The first improved approximation we find as

$$r = r_1 = \xi_1 + \tilde{F}_r(\xi_1,\theta_0,t)\,, \qquad \alpha = \alpha_1 = \theta_0 + \tilde{F}_\theta(\xi_1,\theta_0,t)\,. \tag{47}$$

$$F_1 = F_r(r,\alpha,t) = F_{r0}(r,\alpha) + \sum_{n\neq 0}\{P_n(r,\alpha)\cos nt + Q_n(r,\alpha)\sin nt\}\,, \tag{48}$$

$$\tilde{F}_r(r,\alpha,t) = \sum_{n\neq 0}\frac{1}{n}\{P_n(r,\alpha)\sin nt - Q_n(r,\alpha)\cos nt\}\,, \tag{49}$$

$$F_2 = F_\theta(r,\alpha,t) = F_{\theta 0}(r,\alpha) + \sum_{n\neq 0}\{P_n^*(r,\alpha)\cos nt + Q_n^*(r,\alpha)\sin nt\}\,, \tag{50}$$

$$\tilde{F}_\theta(r,\alpha,t) = \sum_{n\neq 0}\frac{1}{n}\{P_n^*(r,\alpha)\sin nt - Q_n^*(r,\alpha)\cos nt\}\,. \tag{51}$$

Notice that the coefficients to the higher harmonics in $\tilde{F}_r$ and $\tilde{F}_\theta$ become numerically small because of the factor $\frac{1}{n}$.

If we solve Eq. (41) to order ϵ in stead of the system Eqs. (39,40), then we have only one scalar equation to order ϵ.

$$\frac{dr}{d\theta} = \epsilon(r^2\cos^2\theta - 1)r\sin^2\theta \tag{52}$$

$$\frac{d\xi_1}{d\theta} = \epsilon \underset{\theta}{M}\{r^2\cos^2\theta - 1)r\sin^2\theta\}|_{r=\xi_1} \tag{53}$$

$$= \frac{\epsilon}{2}\xi_1(1-\frac{1}{4}\xi_1^2) \tag{54}$$

$$\xi_1(\theta) = \frac{2}{\{1-(1-\frac{4}{a_0^2})e^{-\epsilon\theta}\}^{\frac{1}{2}}} \tag{55}$$

were $\xi_1(0) = a_0$.

We could also obtain this from the other solution since $t = \theta_0 - \theta$

$$\xi_1(\theta_0 - \theta) = \frac{2}{\{1 - (1 - \frac{4}{a_0^2})e^{-\epsilon(\theta_0 - \theta}\}^{\frac{1}{2}}} \tag{56}$$

where $\xi_1(0) = a_0$ *d.v.s.* $\theta = \theta_0$. *We introduce* $u = \theta - \theta_0$ *and find*

$$\xi_1(u) = \frac{2}{\{1 - (1 - \frac{4}{a_0^2})e^{-\epsilon u}\}^{\frac{1}{2}}} \tag{57}$$

Example 2.2 (Phase plane -trajectory) *Find an approximate solution for the phase plane trajectories for the equation*

$$\ddot{x} + \epsilon(|\dot{x}| - 1)\dot{x} + x = 0\,. \tag{58}$$

We find

$$\begin{aligned} \dot{r} &= -\epsilon h(r\cos\theta, r\sin\theta)\sin\theta\,, & (59)\\ \dot{\theta} &= -1 - \epsilon h(r\cos\theta, r\sin\theta)\cos\theta\,, & (60) \end{aligned}$$

$$\frac{dr}{d\theta} = \epsilon h(r\cos\theta, r\sin\theta)\sin\theta + \mathcal{O}(\epsilon^2)\,, \tag{61}$$

$$\begin{aligned} \underset{\theta}{M}\{h(r\cos\theta, r\sin\theta)\sin\theta\} &= \underset{\theta}{M}\{(r|\sin\theta| - 1)r\sin^2\theta\} \\ &= \frac{1}{2\pi}\int_0^{\pi}(r\sin\theta - 1)r sin^2\theta d\theta \\ &\quad + \frac{1}{2\pi}\int_{\pi}^{2\pi}(-r\sin\theta - 1)r sin^2\theta d\theta \end{aligned}$$

substituting $\theta = \pi + u$ *in the last integral shows that these two integrals are equal and we obtain*

$$\begin{aligned} \underset{\theta}{M}\{h(r\cos\theta, r\sin\theta)\sin\theta\} &= \frac{1}{\pi}\int_0^{\pi}(r\sin\theta - 1)r sin^2\theta d\theta \\ &= \frac{1}{\pi}(\frac{4}{3}r^2 - \frac{\pi}{2}r)\,. \end{aligned}$$

The equation for the first approximation becomes

$$\begin{aligned} \frac{d\xi}{d\theta} &= \frac{\epsilon}{\pi}\{\frac{3}{4}\xi^2 - \frac{\pi}{2}\xi\}\,, \\ \frac{d\xi}{d\theta} &= 0 \quad \text{for} \quad \xi = \frac{3}{8}\pi = \xi^* \quad (\text{limitsyklus})\,, \\ \frac{d\xi}{d\theta} &= \frac{4\epsilon\xi}{3\pi}(\xi - \xi^*)\,, \end{aligned}$$

this equation is separable and we find $\xi = \xi(\theta)$ as a first approximation. (See for example Smith & Jones page 110).

The improved first approximation is obtained by taking ξ above and substitute in the following equation

$$x(\theta) = \xi(\theta) + \epsilon \tilde{F}(\xi, \theta)\,, \tag{62}$$

where

$$\begin{aligned} F(x,\theta) &= F_0(x) + \sum_{n\neq 0} F_n e^{in\theta} & (63)\\ \tilde{F}(x,\theta) &= \sum_{n\neq 0} \frac{1}{n} F_n e^{in\theta}\,. & (64) \end{aligned}$$

In our case we find

$$F(x,\theta) \equiv (x|\sin\theta| - 1)x\sin^2\theta\,. \tag{65}$$

Example 2.3 (The inverted pendulum) *It is governed by the equation*

$$\frac{d^2\theta}{d\tau^2} + 2\epsilon\alpha\frac{d\theta}{d\tau} - (k^2\epsilon^2 - \epsilon\sin\tau)\theta = 0\,. \tag{66}$$

We define new variables ϕ and Ω by

$$\begin{aligned} \theta &= \phi + \epsilon\sin\tau\,\phi & (67)\\ \frac{d\theta}{d\tau} &= \epsilon\Omega + \epsilon\cos\tau\,\phi\,. & (68) \end{aligned}$$

We take the derivative of Eq. (67) and substitute in Eq. (68), the result can be written as

$$\dot{\phi} = \frac{\epsilon\Omega}{1+\epsilon\sin\tau}\,. \tag{69}$$

We take the derivative of Eq. (68) and substitute the result in Eq. (66), to obtain

$$\begin{aligned} \ddot{\theta} &= \epsilon\dot{\Omega} - \epsilon\sin\tau\,\phi + \epsilon\cos\tau\,\dot{\phi}\\ &= -2\epsilon\alpha(\epsilon\Omega + \epsilon\cos\tau\,\phi) + (k^2\epsilon^2 - \epsilon\sin\tau)(\phi + \epsilon\sin\tau\,\phi)\\ &= -2\epsilon^2\alpha\Omega - 2\epsilon^2\alpha\phi\cos\tau + (k^2\epsilon^2 - \epsilon\sin\tau)\phi + \epsilon\sin\tau(k^2\epsilon^2 - \epsilon\sin\tau)\phi\,, \end{aligned}$$

$$\begin{aligned} \epsilon\dot{\Omega} + \epsilon\cos\tau\dot{\phi} &= -2\epsilon^2\alpha\Omega - 2\epsilon^2\alpha\phi\cos\tau + k^2\epsilon^2\phi\\ &\quad +\epsilon^3k^2\phi\sin\tau - \epsilon^2\sin^2\tau\,\phi\,, \end{aligned}$$

or

$$\begin{aligned}\dot{\Omega} &= -\dot{\phi}\cos\tau - 2\epsilon\alpha\Omega - 2\epsilon\alpha\phi\cos\tau + k^2\epsilon\phi \\ &\quad +k^2\epsilon^2\phi\sin\tau - \epsilon\sin^2\tau\,\phi\,.\end{aligned}$$

Collecting the results we have

$$\dot{\phi} = \frac{\epsilon\Omega}{1+\epsilon\sin\tau} = \epsilon\Omega - \epsilon^2\Omega\sin\tau + \mathcal{O}(\epsilon^3)\,. \tag{70}$$

$$\begin{aligned}\dot{\Omega} = &- \epsilon\Omega\cos\tau - \epsilon^2\Omega\sin\tau\cos\tau \\ &- 2\epsilon\alpha\Omega - 2\epsilon\alpha\phi\cos\tau + k^2\epsilon\phi\sin\tau \\ &+ k^2\epsilon\phi - \epsilon\sin^2\tau\,\phi + \mathcal{O}(\epsilon^2)\,.\end{aligned}$$

The equations are now in a form appropriate for the averaging method. The averaged equations are:

$$\dot{\phi} = \epsilon\Omega\,, \tag{71}$$

$$\dot{\Omega} = -2\epsilon\alpha\Omega + k^2\epsilon\phi - \frac{1}{2}\epsilon\phi\,, \tag{72}$$

or

$$\ddot{\phi} + 2\epsilon\alpha\dot{\phi} + \epsilon^2(\frac{1}{2} - k^2)\phi = 0\,. \tag{73}$$

The characteristic polynomial is

$$r^2 + 2\epsilon\alpha r + \epsilon^2(\frac{1}{2} - k^2) = 0\,, \tag{74}$$

which have the roots

$$r = -\epsilon\alpha \pm \sqrt{\epsilon^2\alpha^2 - \epsilon^2(\frac{1}{2} - k^2))}\,. \tag{75}$$

We notice that for $\alpha = 0$, we have stable solutions for $k^2 < \frac{1}{2}$. This is a well known result.

3 Dynamic systems

3.1 Introduction

Phenomena developing in time are often modeled in terms of differential equations or systems of such equations. Classic examples are the motion of the earth around the sun, other planets containing moons, meteorites etc. Other examples are the evolution of phenomenon in biology, economy, resource management etc. We distinguish between ordinary (ode) or partial (pde) differential equations. In our context we shall limit ourselves to ode's.

We also distinguish between *linear* and *non linear* equations. The linear equations are already to a certain extent studied in M 117, therefore we shall here be especially concerned with *non linear* equations and their solutions. Another distinction is in between, what is called *autonomous* and *non autonomous* equations or systems of equations. This means systems not being explicitly dependent of time (*autonomous*) and systems that are explicitly dependent of time (*non autonomous*).

3.2 General systems, phase space

Formally we write the equations for such systems as

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)\,, \quad \dot{\mathbf{x}} \stackrel{def}{=} \frac{d\mathbf{x}(t)}{dt}\,, \tag{76}$$

and if time t is not explicitly contained in the lefthand side in this equation then the system is *autonomous* and then we have

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})\,, \tag{77}$$

otherwise the system is called *non autonomous*.

Here we have that $\mathbf{x}(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\}^{\mathrm{t}}$ is a column vector of arbitrary length n. The theory for such systems is well developed and we have existence and uniqueness theorems that guarantee the existence of a unique solution of the initial value problem under given conditions.

The proof for this is given in Appendix A, but only for the one dimensional case. The proof for the n-dimensional case follows the same procedure and is a more or less straight forward generalization.

We notice especially that the condition that has to be satisfied by the function or operator $\mathbf{X}$ is what frequently is referred to as the Lipschitz condition[1].

This condition is somewhat weaker than the requirement that $\mathbf{X}$ should be differentiable. We often say that $\mathbf{x}(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\}^{\mathrm{t}}$, where $\mathbf{x}(0) = \mathbf{x}_0$, represents a solution trajectory in the n-dimensional phase space having the starting point in $\mathbf{x}_0$. The vector $\mathbf{x}(t)$ represents the state of the system at the actual point in time t. But t can also be considered to be a parameter for a curve representation (the trajectory) of the solution. Relative to this trajectory we know from the theory for space curves (generalized to the n-dimensional case) that $\dot{\mathbf{x}}(t)$ can be interpreted as the tangent vector to this curve.

[1]The Lipschitz condition: $\exists A$ such that $|\mathbf{X}(\mathbf{x}_1) - \mathbf{X}(\mathbf{x}_2)| \leq A|\mathbf{x}_1 - \mathbf{x}_2|$, where A is a constant.

Thus the state vector will move as time passes on, along a path (trajectory) that represents the solution curve. The theory for initial value problems also tells us that through every *regular point* in phase space there passes one and only one solution curve (the uniqueness theorem). By a *regular point* we mean a point where the requirements for existence and uniqueness are fulfilled. The fact that two paths (trajectories) in phase space can not cross each other in a regular point is a fundamental property of such systems.

In addition we have especially for autonomous systems that we may choose the starting point in an arbitrary point on a path (trajectory) and this will uniquely determine the whole trajectory (backward and forward).

In this context we also talk about half paths (trajectories) which means that part of the solution curve that starts in the specified point, $\mathbf{x}_0$, and continue in the direction determined by time so that we have a *positive half path* for $t \geq 0$ and a *negative half path* for $t \leq 0$.

Also notice that in principle all systems can be made autonomous by an artificial trick, namely by extending the dimension of a given n-dimensional system to a $(n+1)$-dimensional system and let time t be the new coordinate $t = x_{n+1}$ where $\dot{x}_{n+1} = 1$. The price we have to pay for this is an increase in dimension for our system. This makes the system more complicated to deal with so we often do not achieve any advantage in practical terms when it comes to finding solutions to the problem. However, for general theoretical discussions this may be an interesting way to go.

3.2.1 Existence and uniqueness

Regarding the initial value problem we have the following theorem

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}, t)\,, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{78}$$

where $\mathbf{x}(t)$ and $\mathbf{X}(\mathbf{x}, t)$ are n-dimensional vector functions and $\mathbf{x}_0$ is a given vector which is the starting point. Let $\mathbf{X}(\mathbf{x}, t)$ be continuous on $\bar{G} \times I$, where $I = [t_0,\, t_0 + T]$ and G is open and limited in $\mathbb{R}^n$, such that $\bar{G}$ is compact[2]. We say that $\mathbf{x}(t)$ is a solution of Eq. (78) when a $\mathbf{x}(t)$ is found which have a continuous derivative in a time interval such that $\mathbf{x}(t_0) = \mathbf{x}_0$, and satisfies Eq. (78). We then have the following theorem.

Theorem 3.1 (Existence) *If $\mathbf{X}(\mathbf{x}, t)$ is continuous on $\bar{G} \times I$ where $I = [t_0,\, t_0 + T]$ and G is open and limited in $\mathbb{R}^n$, then for $\forall\, \mathbf{x}_0 \in G\,, \quad \exists\, \mathbf{x}(t)$ with a derivative such that Eq. (78) is satisfied in an interval $t \in [t_0,\, t_0 + h]$ where $h \stackrel{def}{=} \inf(T,\, \frac{d}{M})$ with $M = \sup_{\mathbf{x} \in \bar{G},\, t \in I} ||\mathbf{X}||$ and $d = \operatorname{dist}(\mathbf{x}_0, G^*)$ is the distance from $\mathbf{x}_0$ to the boundary of $\bar{G}$ which is $G^* \stackrel{def}{=} \bar{G} - G$.*

Theorem 3.2 (Uniqueness) *If $\mathbf{X}(\mathbf{x}, t)$ also satisfies the Lipschitz condition:*

$$\|\mathbf{X}(\mathbf{x}_1, t) - \mathbf{X}(\mathbf{x}_2, t)\| \leq k\|\mathbf{x}_1 - \mathbf{x}_2\|\,, \quad \forall\, \mathbf{x}_1,\, \mathbf{x}_2 \in \bar{G}\,, \quad t \in I\,,$$

Then the solution in Theorem 3.1 is unique.

[2] $\bar{G}$ here means the closure of the open set G

Here k is a constant which often is called the Lipschitz constant. The next theorem tells us something about how the solutions in the theorems above, 3.1 and 3.2, can depend on a parameter vector. Given the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}, t, \mathbf{y}) \,, \qquad \mathbf{x}(t_0, \mathbf{y}) = \mathbf{x}_0(\mathbf{y}) \tag{79}$$

where $\mathbf{y}$ is a parameter vector with dimension p.

Theorem 3.3 (Parameter) *Let $\mathbf{X}(\mathbf{x}, t, \mathbf{y})$ be continuous on:*

$$\{\mathbf{x} : \mathbf{x} \in \bar{G}\} \times \{\mathbf{y} : \mathbf{y} \in \bar{K}\} \times \{t : t \in [t_0, t_0 + T]\}$$

were $G \in \mathbb{R}^n$ and $K \in \mathbb{R}^m$ are open and limited, and m and n are arbitrary and determined by the problem. If there exists a constant k such that

$$||\mathbf{X}(\mathbf{x}_1, t, \mathbf{y}) - \mathbf{X}(\mathbf{x}_2, t, \mathbf{y})|| \leq k||\mathbf{x}_1 - \mathbf{x}_2|| \,, \quad \forall\, \mathbf{x}_1,\, \mathbf{x}_2 \in \bar{G}, \quad \mathbf{y} \in \bar{K},\ t \in I \,,$$

then there exists a solution to Eq. (79) which is continuous with respect to the parameter vector $\mathbf{y}$

Notice: Under the same conditions a solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is a continuous function with respect to t_0, $\mathbf{x}_0$. This follows from the proof for existence and uniqueness (this is given in Appendix A, however, only for the one dimensional case. The proof for the general case can be carried through without any essential complications by introducing a convenient norm and distance in the n-dimensional space).

Otherwise the proves for theorem 3.3 and the following theorem 3.4 and 3.5, are not presented here, because there is not time to include them in this course (M214). Interested readers can find more details in standard litterature about this subject, for example [4] and [5].

The next question is: When does an eventual solution have a derivative with respect to the initial conditions. The answer is given in the next theorem:

Theorem 3.4 (Initial conditions) *Given the system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}, t) \,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \,, \ \mathbf{x} \in \bar{G} \in R^n \,, \quad t \in [t_0,\, t_0 + T] \tag{80}$$

where $\mathbf{X}(\mathbf{x}, t)$ is continuous in $\mathbf{x}$ and t and $\frac{\partial \mathbf{X}}{\partial \mathbf{x}}$ exist and is continuous. Then there exist a solution $\mathbf{x}(t)$ on the interval $[t_0,\, t_0 + h]$ which has a continuous first derivative with respect to $\mathbf{x}_0$, where $h \stackrel{def}{=} \inf(T,\, \frac{d}{M})$ with

$$M \stackrel{def}{=} \sup_{\bar{G} \times I} ||\mathbf{X}(\mathbf{x}, t)|| \,, \quad d \stackrel{def}{=} \operatorname{dist}(\mathbf{x}_0,\, G^*) \,, \quad G^* \stackrel{def}{=} \bar{G} - G \,.$$

Notice that the condition required in this theorem, namely that $\frac{\partial \mathbf{X}}{\partial \mathbf{x}}$ exists and is continuous, is more restrictive than the Lipschitz condition. The next theorem tells us how a solution can be extended in time.

Theorem 3.5 (Extension)

(1) Given the autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}) \,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \,. \tag{81}$$

Let $\mathbf{X}(\mathbf{x})$ *be a continuous Lipschitz-function in* $\bar{G} \times [t_0, \, t_0 + T]$. *If* $T \to \infty$ *and* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{c}$, *where* $\mathbf{c}$ *is a constant vector, then* $\mathbf{X}(\mathbf{c}) = \mathbf{0}$

(2) Given the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}, t) \,, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \,. \tag{82}$$

Let $\mathbf{X}(\mathbf{x}, t)$ *be a continuous Lipschitz-function for every* $\mathbf{x}, \; t \geq 0$. *Then we have one of two possible cases:*

(a) There exists a solution $\mathbf{x}(t)$ *for* $t \in [t_0, \, +\infty]$
or

(b) There exists a $T < \infty$ *such that the solution* $\mathbf{x}(t)$ *is defined for* $t \in [t_0, \, T]$ *and* $||\mathbf{x}(t)|| \to \infty$ *when* $t \to T$

3.3 A fluid picture

The way our problem is formulated in Eq. (76), we have that $\mathbf{X}(\mathbf{x}, t)$ are tangent vectors with respect to the solution curves. These solution curves can also be regarded as field lines to the vector field $\mathbf{X}(\mathbf{x}, t)$.

The definition of a field line with respect to the vector field $\mathbf{X}(\mathbf{x}, t)$ is

$$\frac{d\mathbf{x}}{dt} \times \mathbf{X}(\mathbf{x}, t) = \mathbf{0} \,. \tag{83}$$

Thus, the fieldlines are those curves that have the property of being parallel to $\mathbf{X}$ in every point. Therefore the problem of finding the field lines is analogous to solving Eq.(76) for arbitrary initial conditions.

For the non- autonomous system we can think of these field lines as given for a certain value of t. To get a perspective on the problem let us first consider the autonomous system. In principle there is one and only one field line passing through every regular point in the actual phase space, $\mathbf{x}$ -space, and this picture of field lines is constant in time.

This family of curves, the field lines, can be looked upon as analogous to streamlines in a fluid. In hydrodynamics (M 242) one distinguishes between a stationary and a non stationary system. This will in our case correspond to autonomous and non autonomous systems. In analogy to hydrodynamics one may start "many" systems simultaneously with a continuum of starting values, $\mathbf{x}_0$. This way one may picture this as a "streaming" fluid, *phase fluid*.

The solution curves or paths can be looked upon as *stream lines* in this *phase fluid*.

If we picture a small volume element in this fluid moving along, than every point in this volume element follows a distinct path, curve. The volume element, as it appears some time later, at time $t + \delta t$, can be looked upon as a transformation in time. This transformation is governed by the dynamic equations. In general we know from the theory for integrals (M112) that such transformations follow the transformation law

$$dV(t) = JdV(0) \tag{84}$$

where J is the determinant to the Jacobian matrix (dyad) $\mathbf{J}$ with elements J_{ij} where

$$J_{ij} = \frac{\partial x_i(t, \mathbf{x}_0)}{\partial x_{0j}}, \quad \text{and} \quad x_{0j} \stackrel{def}{=} x_j(0, \mathbf{x}_0)\,. \tag{85}$$

Now we look at a volume element that has moved a small interval of time Δt, such that we have

$$\mathbf{x}(\Delta t, \mathbf{x}_0) = \mathbf{x}_0 + \dot{\mathbf{x}}(t, \mathbf{x}_0)_{t=0}\Delta t + \mathcal{O}((\Delta t)^2)$$

and

$$\begin{aligned} \mathbf{J} &= \frac{\partial \mathbf{x}(\Delta t, \mathbf{x}_0)}{\partial \mathbf{x}_0} \\ &= \mathbf{I} + \frac{\partial \dot{\mathbf{x}}(t, \mathbf{x}_0)_{t=0}}{\partial \mathbf{x}_0}\Delta t + \mathcal{O}((\Delta t)^2) \\ &= \mathbf{I} + \frac{\partial \mathbf{X}(\mathbf{x}, t)_{t=0}}{\partial \mathbf{x}_0}\Delta t + \mathcal{O}((\Delta t)^2)\,. \end{aligned}$$

We then find the Jacobian determinant

$$J = \det \mathbf{J} = 1 + \Delta t\, \nabla \cdot \mathbf{X} + \mathcal{O}((\Delta t)^2) \tag{86}$$

where $\nabla \cdot \mathbf{X}$ is[3] evaluated for $t = 0$. Notice that the starting point is chosen arbitrarily. Furthermore we find

$$\frac{dV(\Delta t) - dV(0)}{dV(0)\Delta t} = \nabla \cdot \mathbf{X} + \mathcal{O}(\Delta t)\,, \tag{87}$$

such that the relative growth rate for the volume element becomes

[3]Here ∇ refer to the operator in the n-dimensional $\mathbf{x}$-space.

$$\frac{1}{dV}\frac{d}{dt}dV = \nabla \cdot \mathbf{X}\,, \tag{88}$$

where $\nabla \cdot \mathbf{X}$ is evaluated in the starting point. According to Eq. (76) $\dot{\mathbf{x}} = \mathbf{X}$, we have that $\mathbf{X}$ can be interpreted as the velocity (vector) to the considered volume element. Eq. (88) is then analogous to what is called a material volume element in hydrodynamics. However, in hydrodynamics the volume element in question is 3-dimensional, whereas in our connection this can be a "volume element" of arbitrary dimension.

3.4 Conservative, dissipative systems

A dynamic system can be either *conservative* or *dissipative* dependent on whether the "energy" is conserved or not. Normally a system is *dissipative*, it is processes like friction, heat conduction, viscosity etc. that leads to a *dissipative* system. From our recent finding it is now easy to classify a system according to the criteria

$$\nabla \cdot \mathbf{X} = 0\,, \tag{89}$$

gives a conservative system and

$$\nabla \cdot \mathbf{X} < 0\,, \tag{90}$$

gives a dissipative system. In principle we can imagine a system where the inequality in Eq. (90) is turned around. If this is the case one would have a system being supplied with energy, it is in fact possible to "drive" a system by a periodic force for example like we did for the inverted pendulum. Such a system will receive a net input of energy.

3.4.1 Hamiltonian systems

An important subclass of dynamic systems are they so called Hamiltonian systems, where the governing equations have the following structure:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \tag{91}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \tag{92}$$

here q_i and p_i are generalized coordinates and momentum coordinates. The Hamiltonian function $H = H(q_1, q_2, \ldots q_n, p_1, p_2, \ldots, p_n)$ is initially a given function. In some cases it is related to the energy in the system, but not always. More details about Hamiltonian systems are given in the course M142.

We introduce a $2n \times 2n$ antisymmetric block matrix

$$\mathbf{R} = \left\{ \begin{array}{cc} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{array} \right\}, \tag{93}$$

where $\mathbf{0}$ is a $n \times n$ null matrix and $\mathbf{I}$ is a $n \times n$ unit matrix. Furthermore we use the vector notation

$$\mathbf{x} = \{q_1, q_2, \ldots q_n, p_1, p_2, \ldots, p_n\}^t, \tag{94}$$

which is a column vector. One can then write this system in compressed form as [4]

$$\dot{\mathbf{x}} = \mathbf{R} \cdot \nabla H, \tag{95}$$

where ∇ is an operator now referring to the $2n$- dimensional space defined by the $\mathbf{x}$-vector (Eq. (94)). It is easy to show that $\nabla \cdot \dot{\mathbf{x}} = 0$. The result follows straight forward by using the Hamiltonian equations Eqs. (91) and (92). This shows that

$$\nabla \cdot (\mathbf{R} \cdot \nabla H) = 0,$$

thus, a Hamiltonian system is always conservative.

Example 3.1 (Closed trajectories) *We consider a two-dimensional system and assume there exist solution curves that are closed trajectories. We look at such a trajectory (path) and apply Gauss' theorem in two dimensions*

$$\oint_{\partial S} \mathbf{X} \cdot \mathbf{n} ds = \int \int_S \nabla \cdot \mathbf{X} dx dy, \tag{96}$$

where ∂S is the closed path and $\mathbf{n}$ is a unit normal vector to this path such that $\mathbf{X} \cdot \mathbf{n} = 0$ along the path. We thus have the lefthand side in Eq. (96) equal zero, thus the right hand side must also be zero.

Therfore we can not have closed paths for a nonconservative system for which $\nabla \cdot \mathbf{X}$ has a fixed sign.

The following theorem is important regarding closed trajectories in two dimensions.

Theorem 3.6 (Poincaré - Bendixson) *Let D be a closed limited region in $\mathbb{R}^2$, which do not contain any singular[5] point for the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, which is a 2×2 system. If there exist a positive half path $\mathcal{H}$ which has all the points in D, then three possibilities exist:*

(1) $\mathcal{H}$ is a closed path.

(2) $\mathcal{H}$ approaches a closed path.

(3) $\mathcal{H}$ approaches an equilibrium point (this is really a special case of (2), since an equilibrium point can be considered to be a degenerate case of a closed path).

The proof of this theorem will not be given here, it can be found for example in Strubble [6] page 182.

[4] This is frequently referred to as *symplectic* notation. See for example Goldstein: Classical Mechanics page 347

[5] Singular points here are points where $\mathbf{X}(\mathbf{x}) = \mathbf{0}$, i.e. equilibrium points.