3.5 Equilibrium and stability

In a dynamic system we have by definition equilibrium provided $\dot{\mathbf{x}} = \mathbf{0}$ for all t. In other words: The equilibrium points are those points which satisfies $\mathbf{X}(\mathbf{x}) = \mathbf{0}$ for autonomous systems. In this case equilibrium refers to certain points in phase space which have the property that if the system is located in such a point, it sits there forever, because the derivative of all the coordinates are zero.

A different kind of equilibrium is the one we encounter in the earth motion around the sun. In this case we do not have an equilibrium situation like the one we just described. Still this system is also in some kind of equilibrium in the sense that this system repeats its motion periodically. We have a closed path which after a year repeats itself. It is meaningful to ask the question about stability in both these cases, however, these two phenomenon are so different that they require different concepts of stability in order to catch the important properties of each system.

As a third possibility one can imagine a *motion*, that is a solution curve for Eq. (76) which starts in $\mathbf{x}_0$. What will happen if we start a *motion* close by $\mathbf{x}_0$? Are we then going to find a *motion* which always will be close to the motion that started in $\mathbf{x}_0$. Liapunov has given the name to the stability definition which deals with this problem.

A forth concept of stability is due to Laplace. All *motion* which is limited, that is $|\mathbf{x}(t)| < \infty \quad \forall\, t$ is stable. We are now going to explore these different concepts of stability in more detail.

3.5.1 Stability of an equilibrium

In order to investigate if a given equilibrium is stable or not we can look at what is happening if we start a *motion* near an equilibrium point, $\mathbf{x}_E$. Let the starting point for the *motion* we look at be $\mathbf{x}_0 = \mathbf{x}_E + \delta\mathbf{x}$ and let $\mathbf{x} = \mathbf{x}_E + \boldsymbol{\xi}$ where the *motion* is governed by Eq. (77) (that is we study an autonomous system). We substitute for x in this equation and find

$$\dot{\boldsymbol{\xi}} = \mathbf{X}(\mathbf{x}_E + \boldsymbol{\xi})\,. \tag{97}$$

In most cases it is an impossible task to solve such a problem analytically because the operator $\mathbf{X}$ is nonlinear. In order to make progress we have to make approximations, and look for approximate solutions by linearizing the equation. In practical terms this means that we make a series expansion (in the coordinates) of the right hand side around the equilibrium point. $\mathbf{x}_E$, A Taylor expansion around $\mathbf{x}_E$, results in

$$\dot{\boldsymbol{\xi}} = \mathbf{X}(\mathbf{x}_E) + \boldsymbol{\xi} \cdot \nabla \mathbf{X}(\mathbf{x})_{\mathbf{x}=\mathbf{x}_E} + \mathcal{O}(|\boldsymbol{\xi}|^2)\,. \tag{98}$$

or since $\mathbf{X}(\mathbf{x}_E) = \mathbf{0}$ ($\mathbf{x}_E$ is an equilibrium point), one finds to leading order

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\xi} \cdot \nabla \mathbf{X}(\mathbf{x})_{\mathbf{x}=\mathbf{x}_E}\,, \quad \boldsymbol{\xi}(0) = \delta\mathbf{x}\,, \tag{99}$$

and this is a linear problem that we recognize from M 117. We repeat the results her: It is the eigenvalues λ_n to the matrix $\nabla \mathbf{X}(\mathbf{x})_{\mathbf{x}=\mathbf{x}_E}$ which determine the stability question. If there are n distinct different eigenvalues λ_n and real part, $\Re(\lambda_n) \leq 0$ for all n, then the system is stable, otherwise it is unstable (coinciding eigenvalues needs special consideration. This problem will be discussed in more detail under section **3.5.2** for the special case that we have a two dimensional system.

Notice that for a linear system this coincides with Laplace's definition of stability and with Liapunov's definition that we will discuss shortly. However, it is most natural to look at this problem as a special case of Liapunov stability. See section (**3.7**)

Example 3.2 (The pendulum equation) *The pendulum equation can be written as*

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \tag{100}$$

When we introduce $x = \theta$ and $y = \dot{\theta}$, this equation can be written as a system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2 \sin x \end{pmatrix}. \tag{101}$$

Apparently we have an equilibrium point at $y = 0, \quad x = n\,\pi$ where $n = 0,\, 1,\, \ldots$ We focus on the equilibrium point $(0, 0)$ and linearize the system around this point. By doing so we obtain

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{102}$$

where the matrix $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$ have the eigenvalues $\lambda_\pm = \pm i\omega$, and as expected the system is stable.

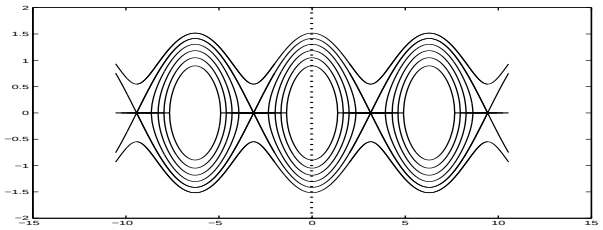


Figure 1: *Phase plan plots of the pendulum equation. The horizontal axis is $x = \theta$ and the vertical is $y = \dot{\theta}$).*

3.5.2 Classifying equilibria in two dimensions

For two dimensional systems Eq. (99) can be written as

$$\dot{x} = ax + by\,,$$
$$\dot{y} = cx + dy\,.$$

where $\boldsymbol{\xi} \stackrel{def}{=} \{x, y\}^{\mathrm{t}}$ and the matrix

$$\nabla \mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\mathbf{def}}{=} \mathbf{A}\,. \tag{103}$$

This is a simple linear system of first order ode's. For finding a solution, the usual procedure is to assume the following form

$$\mathbf{x} = \mathbf{x}_0\, e^{\lambda t}\,.$$

where $\mathbf{x} \stackrel{def}{=} \{x, y\}^{\mathrm{t}}$ and $\mathbf{x}_0$ is the initial value for $\mathbf{x}$. We then find

$$\dot{\mathbf{x}} = \lambda\, \mathbf{x}$$

thus this problem is reduced to finding the solution to the algebraic problem

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}\,, \tag{104}$$

that is an eigenvalue problem for the matrix $\mathbf{A}$. The solutions will now get the properties determined by the eigenvalues that can be real or complex with a real part that can be positive or negative. In general the solutions will have the following form:

$$\mathbf{x}(t) = c_1 \mathbf{e} e^{\alpha t + i\beta t} + c_2 \mathbf{e}^* e^{\alpha t - i\beta t} \tag{105}$$

where c_1 and c_2 are integration constants, $\mathbf{e}$ is the eigenvector corresponding to the eigenvalue $\lambda = \alpha + i\beta$ and $\mathbf{e}^*$ is the complex conjugate of $\mathbf{e}$. Notice that the condition for having a real solution when λ is complex is $c_2 = c_1^*$. In general we have that a complex eigenvalue implies that we also have the complex conjugate of this eigenvalue as an eigenvalue since the matrix $\mathbf{A}$ is real. The sign of α determines the stability property. In the case of complex eigenvalues we notice that α is the same for both eigenvalues. However, in the case of real eigenvalues the sign can be different for the two eigenvalues. As already mentioned, in general we have the following result:

If there exists an eigenvalue which is positive or having positive real part, then the system is unstable otherwise it is stable.

This corresponds to defining solutions that are limited as stable and those that are unlimited as unstable (Laplace definition of stability). Later we shall see that this is also in agreement with Liapunov's definition of stability.

The eigenvalue equation is the following determinant put equal to zero

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (a + d)\lambda + ad - bc = 0\,. \tag{106}$$

Let $2p \stackrel{def}{=} (a + d)$ and $q \stackrel{def}{=} \det(\mathbf{A}) = ad - bc$, we can then write the eigenvalue equation as $\lambda^2 - 2p\lambda + q = 0$. The roots of this equation are

$$\lambda_{\pm} = p \pm \sqrt{\Delta}\,, \tag{107}$$

where $\Delta \stackrel{def}{=} p^2 - q$, which often is called the discriminant for the system. It is the discriminant that determines what kind of solutions we have. We have that $\Delta < 0$ gives complex solutions, $\Delta = 0$ gives coinciding real roots with the corresponding polynomial solution and finally $\Delta > 0$ gives two real roots. When the roots are complex the stability is determined by the sign of p. When the roots are real it gets more complicated.

The case $\Delta > 0,\ \ q > 0$

Let the eigenvector $\mathbf{e} \stackrel{def}{=} \{r, s\}^{\mathrm{t}}$. When $\lambda = \lambda_+$ we find these equations for r and s:

$$\begin{aligned} (a - \lambda_+)r + bs &= 0\,, \\ cr + (d - \lambda_+)s &= 0\,. \end{aligned}$$

Since the determinant to this system of equations is zero the equations must be dependent and we can chose for example $r = r+$ freely and use the other equation to determined s_+. From this procedure we obtain $r = r_+$ and $s = s_+$ such that

$$\mathbf{e}_+ = \begin{bmatrix} r_+ \\ s_+ \end{bmatrix} \neq \mathbf{0}\,.$$

The corresponding solutions for $\lambda = \lambda_-$ is then obtained, and the general solution for the system can be written as

$$\mathbf{x}(t) = c_1 \mathbf{e}_+ e^{t\lambda_+} + c_2 \mathbf{e}_- e^{t\lambda_-}$$

where c_1 and c_2 are arbitrary integration constants.

Since according to assumption we have $q > 0$, both roots have the same sign and we get what is called a **node**. A **node** is stable when $p < 0$ and unstable when $p > 0$, since this parameter determines the sign of the roots, that is the eigenvalues $\lambda_{\pm}$. A simple example here is when we in Eq. (106) put: $a = 2,\ b = 0,\ c = 0,\ d = 1$ (unstable), and $a = -1,\ b = 0,\ c = 0,\ d = -2$ (stable). This is illustrated in Fig. 2 (a).

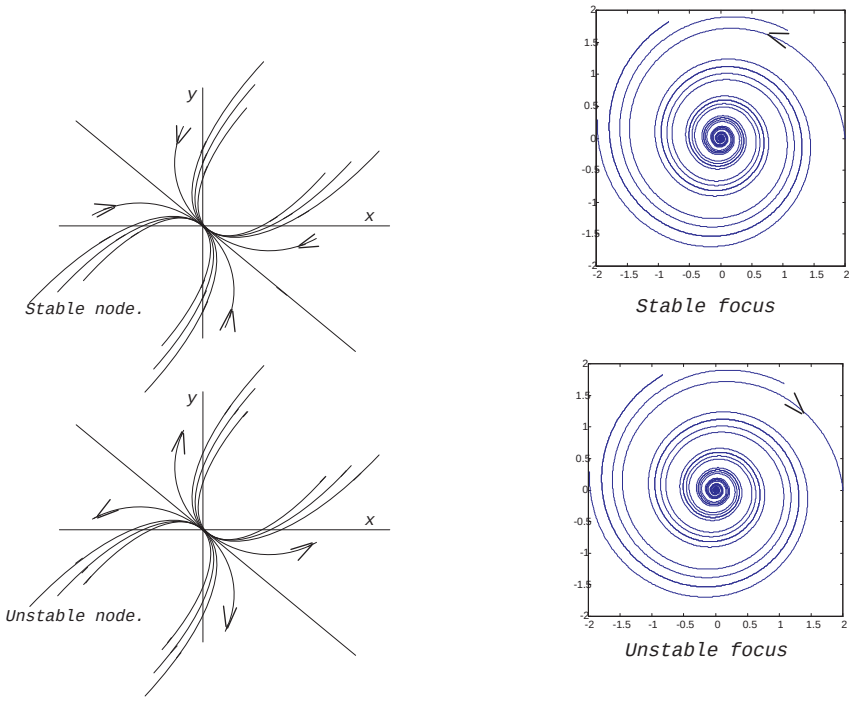
The case $\Delta < 0,\ \ q > 0$

For this case we have a complex pare of roots

$$\lambda = \alpha \pm i\beta\,.$$

In this case the solution is a spiral or a focus which can be either stable or unstable, ($p < 0$, stable), ($p > 0$, unstable). A typical phase plan portrait is shown in Fig. 2 (b).

The case $\Delta > 0,\ \ q < 0$



(a) Node (b) Focus

Figure 2: Stability for 2d-system. The to main-types of equilibrium points **node** and **focus** or **spiral**

In this case the discussion follows the case above, but now the two roots differ in sign, we obtain what is called a **saddle-point**. The phase plane portrait of this case is illustrated in Fig. 3 (a)

The case $\Delta < 0, \;\; q > 0 \; p = 0$

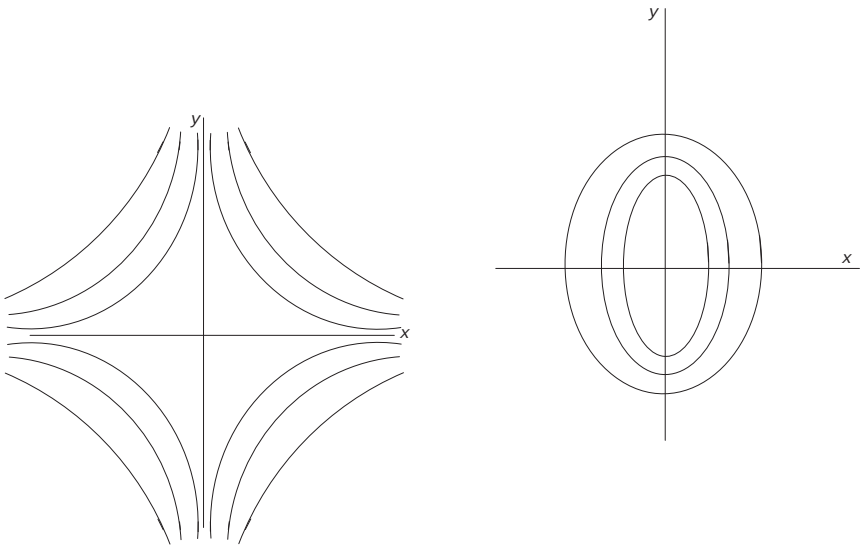
Essentially this is a special case of a focus, but in this case we have no growth or damping in the system. We obtain a purely periodic solution that we call a **center**. See Fig. 3 (b)

The case $\Delta = 0, \;\; p > 0$ The solution is

$$\mathbf{x}(t) = (c_1 + c_2 t)e^{pt}\,, \tag{108}$$

thus for $p > 0$ we have unstable solutions (a degenerate node), and $p < 0$ gives stable solutions (also a degenerate node).

The classification of equilibrium points in two dimensions can now be collected in the following table



(a) Saddle point (b) Center

Figure 3: Stability of 2d-system. A **saddle point** and a **center**

Type	$2p = a + d$	$q = ad - bc$	$\Delta = p^2 - q$
saddle point	-	$q < 0$	$\Delta > 0$
stable node	$p < 0$	$q > 0$	$\Delta > 0$
stable spiral	$p < 0$	$q > 0$	$\Delta < 0$
unstable node	$p > 0$	$q > 0$	$\Delta > 0$
unstable spiral	$p > 0$	$q > 0$	$\Delta < 0$
center	$p = 0$	$q > 0$	$\Delta < 0$
deg. stab. node	$p < 0$	$q > 0$	$\Delta = 0$
deg. unstab. node	$p > 0$	$q > 0$	$\Delta = 0$

It is important to notice that the equilibrium point is normally not part of the trajectory (paths) that leads to or emerge from an equilibrium point. Explain why it must be like this!

Comment: This discussion of a two dimensional system is possible to do in an elegant and transparent way by using the phase plane. The drawback is that a corresponding discussion in higher dimensions is impossible to carry through, things get far too complicate already at three dimensions.

However, some help and insight from two dimensions can also be some kind of guideline for higher dimensions by projecting from a higher dimensional space into the two dimensional space. But this is not a straight forward matter.

3.6 Orbital stability

Orbital stability or Poincarè stability is, as the name suggests a question about whether an orbit in a system is stable or not. This refers to an autonomous system and is of special interest in connection with the motion of the planets around the sun, and similar problems.

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})\,. \tag{109}$$

Let $\mathbf{x}^*(t)$ be the orbit (path) or the solution we want to investigate. We assume the orbit (path) to start at the point $\mathbf{a}^*$ such that $\mathbf{x}^*(0) = \mathbf{a}^*$.

Then consider a region around the starting point $\mathbf{a}^*$ where we can chose a new starting point $\mathbf{a}$, requiring $\|\mathbf{a} - \mathbf{a}^*\| < \delta$. The half path originating in the starting point $\mathbf{a}^*$ we call $\mathcal{H}^*$, and the path originating in $\mathbf{a}$ we call $\mathcal{H}$. We formally define orbital (path) stability by:

Definition 3.1 (Orbital or Poincaré-stability) *A half path $\mathcal{H}^*$ governed by Eq. (109) is stable, if for an arbitrary ϵ there exists a δ, such that*

$$\|\mathbf{a} - \mathbf{a}^*\| < \delta \quad \Rightarrow \quad \sup_{\mathbf{x}\in\mathcal{H}} \operatorname{dist}(\mathbf{x}, \mathcal{H}^*) < \epsilon\,. \tag{110}$$

Here "dist", means the distance form $\mathbf{x}$ to a curve K, defined by:

$$\operatorname{dist}(\mathbf{x}, K) \stackrel{def}{=} \inf_{\mathbf{y}\in K} \|\mathbf{x} - \mathbf{y}\|\,. \tag{111}$$

Notice that a half path can be a periodic motion that repeats itself all the time. A more visual way to formulate this definition is to imagine in three dimensions a path that is embedded in a thin tube having radius ϵ (ϵ chosen arbitrarily). Let us mark a point P where this half path starts. Then let a point Q be arbitrarily chosen within a small "sphere" of radius δ centered in P. Then if a δ can be found such that all half paths that start inside this small δ-sphere remain inside the ϵ-tube for all time, then the path is stable.

Notice that this does not mean that even though the moving points start in P and Q, and initially are close together (inside the δ-sphere) it is not required that they stay close as the motion proceeds, it is only required that the two paths traced by the two particles stay close. It is easy to imagine a periodic motion where two points move along closed paths which are close together even though the points move apart. This would happen if the period for one *motion* is slightly different from the period in the other *motion*.

3.7 Liapunov stability

Definition 3.2 (Liapunov stability) *(i) We now study the general case of a motion (path)* $\mathbf{x}(t)$ *that starts in* $\mathbf{x}^*(t_0) = \mathbf{x}_0^*$*, and look simultaneously at another motion (path) which starts in* $\mathbf{x}(t_0) = \mathbf{x}_0$*.*

If for arbitrary ϵ*, there exists a* $\delta(\epsilon, t_0)$*, such that*

$$\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta \quad \Rightarrow \quad \|\mathbf{x}(t) - \mathbf{x}^*(t)\| < \epsilon \quad \text{for} \quad t > t_0\,, \tag{112}$$

then the motion $\mathbf{x}(t)$ *is stable.*

(ii) **Uniform stability**
If a solution, (motion) $\mathbf{x}^*(t)$*, is Liapunov-stable for* $t \geq t_0$ *and* δ *as given in the definition of Liapunov-stability, (i), is independent of* t_0*, then the solution is uniformly stable for* $t \geq t_0$ *.*

(iii) **Asymptotic stability**
If there under (i) in addition the following is satisfied

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}^*(t)\| = 0,$$

then the system is asymptotically stable.

(iv) One can show, Cesari [4] 1971 page 5: That if a system is Liapunov-stable for starting point t_0*, then it is also Liapunov-stable for the starting point* $t_1 > t_0$*. (This result is almost self evident.)*

(v) **Equilibrium**

It is alright that $\mathbf{x}(t)$ *under (i) is an equilibrium point. An equilibrium point can be considered to be a degenerate case of a path or motion . Therefore the definitions also apply for this case. (confer what was said at the end of section (3.5.1))*

(vi) **Unstable motion**

If $\mathbf{x}^*$ *above do not satisfy the conditions under (i) - (iii), then the system is not (uniformly, asymptotic or otherwise) Liapunov stable.*

For the case where the path degenerates to a point, case (v), this definition of stability is still valid. As mentioned under the introduction to the subject of equilibrium and stability, when we deal with linear systems the Liapunov and Laplace definitions of stability will coincide. This is because for a linear system one can always by multiplying by a constant scale the solution so that it will satisfy the Liapunov condition for stability as long as the solution is limited (Laplace definition).

3.7.1 Liapunov-exponents

We introduce a new concept , *Liapunov-exponents*, as an aid to characterize solutions, especially those which are unstable. Our system as before is given by Eq. (78). We study two solutions, one starting in $\mathbf{x}_0$ and one starting in $\mathbf{x}_0 + \mathbf{y}$ at time $t = 0$. We assume $|\mathbf{y}| \leq \epsilon$, where ϵ is a small parameter and $\mathbf{y}$ is an arbitrary vector. In other words we look at a particular solution $\mathbf{x}(t, \mathbf{x}_0)$, and all solutions starting inside an ϵ-sphere centered at $\mathbf{x}_0$. We shall be interested in the distance between the two solutions defined as $\mathbf{d}(t) \stackrel{def}{=} \mathbf{x}(t, \mathbf{x}_0 + \mathbf{y}) - \mathbf{x}(t, \mathbf{x}_0)$. By series expansion one finds

$$\begin{aligned} \mathbf{d}(t) &\stackrel{def}{=} \mathbf{x}(t, \mathbf{x}_0 + \mathbf{y}) - \mathbf{x}(t, \mathbf{x}_0) \\ &= \mathbf{y} \cdot \frac{\partial \mathbf{x}(t, \mathbf{x}_0)}{\partial \mathbf{x}_0} + \mathcal{O}(\epsilon^2) \\ &= \mathbf{y} \cdot \mathbf{J}(t) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{113}$$

where $\mathbf{J}(t)$ is the Jacobian matrix for the transformation $\mathbf{x}_0 \rightarrow \mathbf{x}(t, \mathbf{x}_0)$. We assume ϵ to be small enough so that second order terms in ϵ can be omitted. This leeds us to the linear transformation

$$\mathbf{d}(t) = \mathbf{y} \cdot \mathbf{J}(t)\,, \tag{114}$$

where the transformation matrix is derived from the original solution $\mathbf{x}(t, \mathbf{x}_0)$. From this it also follows that $\mathbf{d}(0) = \mathbf{y}$, (notice that $\mathbf{J}(0) = \mathbf{I}$, the unit matrix). The matrix $\mathbf{J}(t)$ can not be singular ($\det(\mathbf{J}) \neq 0$), why? Remember the uniqueness theorem. We have an n-dimensional ϵ-sphere that is being mapped on to a new "volume" which in this case must be a n-dimensional "ellipsoid" having semi axes ϵ_1, ϵ_2, ... ϵ_n. Notice here that the only closed surface of degree 2 is an ellipsoid, having a sphere as a special case. Now we scale the problem such that

$$\epsilon_i(t) \stackrel{def}{=} \epsilon\, a_i(t)\,, \tag{115}$$

that is $\epsilon_i(0) = \epsilon$ and $a_i(0) = 1$ for all i. The surface of a sphere with radius ϵ can be written as

$$\mathbf{r} \cdot \mathbf{I} \cdot \tilde{\mathbf{r}} = \epsilon^2\,, \tag{116}$$

where $\mathbf{I}$ is the unit matrix. We use the transformation (114) above and obtain

$$\mathbf{y} \cdot \mathbf{J}(t) \cdot \mathbf{I} \cdot \tilde{\mathbf{J}}(t) \cdot \tilde{\mathbf{y}} = \epsilon^2\,, \tag{117}$$

or

$$\mathbf{y} \cdot \mathbf{J}(t) \cdot \tilde{\mathbf{J}}(t) \cdot \tilde{\mathbf{y}} = \epsilon^2\,, \tag{118}$$

where $\tilde{\mathbf{J}}$ is the transposed of the matrix $\mathbf{J}$. Let λ_i' be the eigenvalues of the symmetric matrix $\mathbf{J}(t) \cdot \tilde{\mathbf{J}}(t)$. We may then write Eq. (118) as

$$y_1^2\lambda_1' + y_2^2\lambda_2' + \ldots + y_n^2\lambda_n' = \epsilon^2\,, \tag{119}$$

or

$$\frac{y_1^2}{\epsilon_1^2(t)} + \frac{y_2^2}{\epsilon_2^2(t)} + \ldots + \frac{y_n^2}{\epsilon_n^2(t)} = 1\,, \tag{120}$$

where $\epsilon_i^2(t) \stackrel{def}{=} \epsilon^2\frac{1}{\lambda_i'}$ and $\epsilon_i(t)$ are the semi axes in the ellipsoid. By defintion we have $a_i(t) = \frac{\epsilon_i(t)}{\epsilon}$. We introduce the notation $\nu_i(t) \stackrel{def}{=} \frac{\ln a_i(t)}{t}$ or $a_i(t) = \exp(t\,\nu_i(t))$ such that for $\nu_i(t)$ constant, $a(t)$ has an exponential growth.

Definition 3.3 (The Liapunov-exponent) *The Liapunov-exponent λ_i, corresponding to $a_i(t)$ is now defined by*

$$\lambda_i \stackrel{def}{=} \lim_{t\to\infty}\nu_i(t) = \lim_{t\to\infty}\frac{\ln a_i(t)}{t}\,. \tag{121}$$

From this definition we can interpret the Liapunov-exponent as an average long term growth rate for the semi axis in the ellipsoid that was our starting point. We observe that the volume of an ellipsoid is proportional to the product of the semi axis. By using Eq. (88), we then find

$$\frac{d}{dt}\ln(\epsilon_1\cdot\epsilon_2\cdots\epsilon_n) = \frac{1}{dV}\frac{d(dV)}{dt} = \nabla\cdot\mathbf{X}\,, \tag{122}$$

and it follows that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \lim_{t\to\infty}\frac{1}{t}\int_0^t \nabla\cdot\mathbf{X}dt\,, \tag{123}$$

notice that $\ln\epsilon_i(t) = \ln a_i(t) +$ constant, such that the sum of Liapunov-exponents becomes the long term mean value of this divergence.

3.7.2 A special class of linear systems (I)

We consider the system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \tag{124}$$

where $\mathbf{x}$ is an n-dimensional vector and $\mathbf{A}(t)$ is an $n\times n$ matrix. We want to study the stability properties with respect to an arbitrary solution $\mathbf{x}^*(t)$. Let $\mathbf{x}(t)$ be some other solution. We define $\boldsymbol{\xi}(t) \stackrel{def}{=} \mathbf{x}(t) - \mathbf{x}^*(t)$, such that $\|\boldsymbol{\xi}(t)\|$ is a measure of the distance between these solutions. The solutions start with a distance $\|\boldsymbol{\xi}(t_0)\|$, where $\boldsymbol{\xi}(t_0) = \mathbf{x}(t_0) - \mathbf{x}^*(t_0)$. We obtain the following equation for $\boldsymbol{\xi}(t)$

$$\dot{\boldsymbol{\xi}}(t) = \mathbf{A}(t)\boldsymbol{\xi}(t)\,. \tag{125}$$

This shows that the stability question, according to Liapunov, for the *null*-solution of this system is the same as for Eq.(124). In this connection we call $\boldsymbol{\xi}(t)$ for a perturbation of the solution $\mathbf{x}^*(t)$.

3.7.3 Structure of a n-dimensional linear system

We have the system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}\,, \tag{126}$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix with elements $a_{ij}(t)$ which are functions of t and furthermore $\mathbf{x}(t)$ is an n-dimensional column vector. The system apparently is not autonomous. Let $\mathbf{x}_i$ where $i = 1, 2, 3, \ldots, n$ be n real or complex solution vectors to Eq. (126). Then we also have that $\mathbf{x}(t) = \sum_i^n \alpha_i \mathbf{x}_i(t)$ is a solution for arbitrary choice of the constants α_i.

Definition 3.4 (Linearly independent vector functions) *Let* $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$, $\ldots, \mathbf{v}_n(t)$ *be real or complex vector functions. If there exist constants* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *not all being zero and such that* $\sum_{i=1}^n \alpha_i \mathbf{v}_i(t) \equiv \mathbf{0}$*, then the vector functions* $\mathbf{v}_1(t), \mathbf{v}_2(t), \ldots, \mathbf{v}_n(t)$ *are linearly dependent, otherwise they are linearly independent.*

Theorem 3.7 (Solution space-1) *All* $n+1$ *systems of solutions of the* n*-dimensional equation (126) are linearly dependent.*

Theorem 3.8 (Solution space-2) *There always exist* n *linearly independent solutions of Eq. (126).*

Theorem 3.9 (Solution space-3) *Let* $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)$ *be an arbitrary set of* n *linearly independent solution vectors (real or complex) of Eq.(126). Then any solution can be written as a linear combination of these solution vectors.*

The proves for these theorems are left for the reader to do.

Definition 3.5 (The fundamental matrix) *Let* $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)$ *be* n *linearly independent solution vectors (real or complex) of Eq. (126). The matrix which is obtained by placing these column vectors side by side*

$$\boldsymbol{\Phi}(t) = (\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)) = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix} \tag{127}$$

we call a fundamental matrix.

Theorem 3.10 (Solutions - solution matrix) *Given an arbitrary solution matrix* $\boldsymbol{\Phi}(t)$ *for Eq. (126). Then either:*

(i) $\det\{\boldsymbol{\Phi}(t)\} = 0, \qquad \forall\, t$*, or*

(ii) $\det\{\boldsymbol{\Phi}(t)\} \neq 0, \qquad \forall\, t$

Notice that the determinant, $\det(\boldsymbol{\Phi}(t))$, is often called the Wronskian. As should be well known, it has the important property that either it is identically equal to zero or it is non zero for all t. This is in correspondence with the results just stated.

Case (i) is what happens when the solutions are linearly dependent.

Case (ii) is what happens when the solutions are linearly independent, and then the solution matrix is a fundamental matrix for the problem. Otherwise see M117 for more details, Boyce and DiPrima [7].

Theorem 3.11 (Solution by the fundamental matrix-1) *The solution of Eq. (126) which satisfies the initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$ *is given as* $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0$*, where* $\mathbf{\Phi}(t)$ *is an arbitrary fundamental matrix.*

Theorem 3.12 (Solution by the fundamental matrix-2a) *The solution of* $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ *which satisfies the initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$ *can be written as*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{f}(s)ds\,, \tag{128}$$

where $\mathbf{\Phi}(t)$ *is an arbitrary fundamental matrix which satisfies* $\dot{\mathbf{\Phi}} = \mathbf{A}(t)\mathbf{\Phi}$.

Proof: This can be shown by taking the derivative of Eq. (128) and using that $\dot{\mathbf{\Phi}}(t) = \mathbf{A}(t)\mathbf{\Phi}(t)$.

Alternatively we also have, when $\mathbf{A}$ in Eq. (124) is a constant matrix:

Theorem 3.13 (Solution by the fundamental matrix-2b) *The solution of* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ *which satisfies the initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$ *can be written as*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0)\mathbf{f}(s)ds\,, \tag{129}$$

where $\mathbf{\Phi}(t)$ *is an arbitrary fundamental matrix which satisfies* $\dot{\mathbf{\Phi}} = \mathbf{A}\mathbf{\Phi}$ *(notice that* $\mathbf{A}$ *is a constant in this case).*

Proof: This can be shown in the same manner as in previous theorem by taking the derivative and using that $\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t)$. Notice that the derivative of the integral contains two contributions, as function of the upper limit and as explicit function of t under the integral sign, and since $\mathbf{A}$ is constant it can be moved outside the integral.

The details are left to the reader to carry through.

Important remark: It is usual and convenient to chose the *fundamental matrix* such that $\mathbf{\Phi}(t_0) = \mathbf{I}$, that is the unit matrix. The formulae simplify by such a choice.

Theorem 3.14 (Linear stability) *Null (and other solutions) of* $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ *for* $t > t_0$ *(where* t_0 *is arbitrary) is stable (Liapunov) if and only if all solutions are limited.*

If $\mathbf{A}$ *is a constant matrix and all solutions are limited, then the solutions are uniformly stable.*

Proof: When all solutions $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0$ are limited, then the fundamental matrix $\mathbf{\Phi}$ must also be limited, that is: There exists a constant M, such that $\|\mathbf{\Phi}\| < M$, where we can have $\|\mathbf{\Phi}\| \stackrel{def}{=} \sum_{i,j} \sqrt{\Phi ij^2}$ [6]. Then let $\mathbf{x}(t)$ be a solution of $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{t})\mathbf{x}$ and given $\epsilon > 0$ arbitrarily small. Then we find

$$\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq \|\mathbf{\Phi}(t)\|\|\mathbf{x}_{01} - \mathbf{x}_{02}\| \leq M\|\mathbf{x}_{01} - \mathbf{x}_{02}\| < \epsilon \tag{130}$$

for $\|\mathbf{x}_{01} - \mathbf{x}_{02}| < \delta = \frac{\epsilon}{M}$. This implies stability in the meaning of Liapunov.

If $\mathbf{A}$ is constant and the system is stable, then the stability property is related to the eigenvalues of $\mathbf{A}$, and this problem is not dependent on t. Thus if such a system is stable then it is uniformly stable.

Q.E.D.

More extensive discussions of systems where $\mathbf{A}$ is a constant matrix can be found in M 117. It could be wise to update yourself on this material!

3.7.4 A special class of linear systems II

At this point we shall study systems of the following form

$$\dot{\mathbf{x}} = \{\mathbf{A} + \mathbf{C}(t)\}\mathbf{x}\,, \tag{131}$$

where $\mathbf{A}$ is a constant matrix. It can be shown quite generally that the stability for such systems is determined by the stability properties of the linear approximation to this system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\,. \tag{132}$$

In order to prove this in a simple way we need a well known lemma called *Gronwall's lemma*. This lemma can be formulated as:

Theorem 3.15 (Gronwall's-lemma) *If we for* $t \geq t_0$ *have satisfied:*

1. $u(t)$ *and* $v(t)$ *are continuous and* $u(t) > 0$ *and* $v(t) > 0$*;*

2.
$$u(t) \leq K + \int_{t_0}^{t} u(s)\, v(s)\, ds\,, \quad \text{where} \quad K > 0\,; \tag{133}$$

then it follows that

$$u(t) \leq K e^{\int_{t_0}^{t} v(s)\, ds}\,, \quad t \geq t_0 \quad K > 0 \tag{134}$$

Proof: The right hand side in Eq. (133) is positive since $K > 0$ and $u(t),\ v(t) \geq 0$. Eq. (133) therfore implies

[6] Notice that there exist different ways to define the norm of a matrix.

$$\frac{u(t)\,v(t)}{K+\int_{t_0}^{t} u(s)\,v(s)\,ds} \leq v(t)\,. \tag{135}$$

By integrating Eq. (135) from t_0 to t we find

$$\ln\left\{K+\int_{t_0}^{t} u(s)\,v(s)\,ds\right\} - \ln K \leq \int_{t0}^{t} v(s)ds\,, \tag{136}$$

or

$$K+\int_{t_0}^{t} u(s)\,v(s)\,ds \leq Ke^{\int_{t0}^{t} v(s)ds}\,. \tag{137}$$

Then if we use Eq. (133) once more, the result given by Eq. (134) is obtained. Q.E.D.

We now use this theorem to prove the following results

Theorem 3.16 *If*

1. $\mathbf{A}$ *is a constant matrix where all the eigenvalues have negative real parts ;*
2. $\mathbf{C}(t)$ *is a continuous matrix for* $t \geq t_0$ *and*

$$\int_{t_0}^{t} \|\mathbf{C}(t)\|dt \quad \textit{are limited for} \quad t > t_0\,, \tag{138}$$

then all the solutions of the system

$$\dot{\mathbf{x}} = \{\mathbf{A}+\mathbf{C}(t)\}\mathbf{x}\,, \tag{139}$$

are asymptotically stable.

Proof: We write the system as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}+\mathbf{C}(t)\mathbf{x}\,. \tag{140}$$

If $\mathbf{x}(t)$ is a solution, then the last term, $\mathbf{C}(t)\mathbf{x}$ can play the role as $\mathbf{f}(t)$ in Eq. (124) where it was proven that the stability properties of the zero solution was equivalent to the stability properties of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Therefore we have from Eq. (139) that

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0)\mathbf{C}(s)\mathbf{x}(s)ds\,, \tag{141}$$

where $\mathbf{\Phi}(t)$ is a fundamental matrix for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. That Eq. (141) is correct can be proved by taking the derivative. But remember to take the derivative both with respect to the upper limit and to the explicit time dependency under the integral sign and in addition make us of the relation $\dot{\mathbf{\Phi}} = \mathbf{A}\mathbf{\Phi}$.

See also theorem **3.12**. Notice that $\mathbf{x}(t_0) = \mathbf{x}_0$. By using the properties of norms and Eq. (141), it can furthermore be shown that

$$\|\mathbf{x}(t)\| \leq \|\mathbf{\Phi}(t)\|\,\|\mathbf{\Phi}^{-1}(t_0)\|\,\|\mathbf{x}_0\| + \|\mathbf{\Phi}^{-1}(t_0)\| \int_{t_0}^{t} \|\mathbf{\Phi}(t-s+t_0)\|\,\|\mathbf{C}(s)\|\,\|\mathbf{x}(s)\| ds\,. \tag{142}$$

We now use that the fundamental matrix $\mathbf{\Phi}(t)$ is related to the matrix $\mathbf{A}$ through the eigenvalues. Since all the solution vectors building the matrix $\mathbf{\Phi}(t)$ are asymptotically stable because of the assumption, all eigenvalues have negative real parts, then there exist two positive numbers, M and m such that

$$\|\mathbf{\Phi}(t)\| \leq M\, e^{-mt}\,, \quad t \geq t_0\,, \tag{143}$$

and from the properties of the fundamental matrix $\Psi(t)$, there exists a number β such that

$$\|\mathbf{\Phi}^{-1}(t_0)\| \leq \beta\,. \tag{144}$$

By using Eq. (143) and Eq. (144), we may rewrite Eq. (142), to obtain

$$\|\mathbf{x}(t)\| e^{mt} \leq M\beta\|\mathbf{x}_0\| + \int_{t0}^{t} \{\|\mathbf{x}(s)\|\, e^{ms}\}\{\|\mathbf{C}(s)\|\, M\,\beta\, e^{-mt_0}\} ds\,. \tag{145}$$

In Eq. (133) we let

$$u(t) = \|\mathbf{x}(t)\|\, e^{mt}\,, \quad v(t) = \|\mathbf{C}(t)\|\,\beta\, M\, e^{-mt_0} \quad \text{and} \quad K = M\,\beta\,\|\mathbf{x}_0\|\,,$$

such that finally we obtain by the aid of the Gronwall's lemma (theorem **3.15**),

$$\|\mathbf{x}(t)\| e^{mt} \leq M\beta\|\mathbf{x}_0\| e^{\beta\, M\, e^{-mt_0} \int_{t_0}^{t} \|\mathbf{C}(s)\| ds} \tag{146}$$

or

$$\|\mathbf{x}(t)\| \leq M\beta\|\mathbf{x}_0\| e^{\{\beta\, M\, e^{-mt_0} \int_{t_0}^{t} \|\mathbf{C}(s)\| ds\} - mt}\,. \tag{147}$$

From this result we see that all solutions are limited for $t > t_0$. Thus they are stable. However, since all solutions also approach zero for $t \to \infty$, they are also asymptotically stable.

If $\mathbf{C}(t)$ satisfies the conditions in theorem **3.16**, but all the solutions to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are not asymptotically stable, only limited. Then the same is valid for the system $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{C}(t)\,\mathbf{x}$. This can be seen by choosing $m = 0$ in Eq. (147).

3.8 Linear systems, periodic coefficients

We shall consider systems of the following type

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}\,, \quad \text{with} \quad \mathbf{P}(t+T) = \mathbf{P}(t)\,, \quad -\infty < t < \infty\,, \tag{148}$$

where $\mathbf{P}(t)$ is an $n \times n$ matrix with elements which are periodic functions of t. The smallest period is T, but the system naturally also contains the periods $2T$, $3T$, $+\ldots$. We find $\nabla \cdot (\mathbf{P}(t) \cdot \mathbf{x}) = \sum_{i,j} P_{ij}\delta_{ij} = \sum_i P_{ii} = \mathrm{tr}\,(\mathbf{P})$, it is therefore the trace in the $\mathbf{P}$-matrix that determines whether the system is conservative or not.

The following example shows that the solutions of such systems are not necessarily periodic:

$$\dot{x} = (1 + \sin t)x\,,$$

having the solution

$$x = ce^{t-\cos t}\,,$$

where c is an arbitrary constant. For the system (148) there exists a theorem (Floquet's theorem):

Theorem 3.17 (Periodic systems) *The regular system* $\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$ *where* $\mathbf{P}$ *is an* $n \times n$ *matrix function with the smallest period* T*, has at least one non trivial solution* $\mathbf{x} = \boldsymbol{\chi}(t)$ *such that*

$$\boldsymbol{\chi}(t+T) = \mu\,\boldsymbol{\chi}(t)\,, \quad -\infty < t < \infty\,, \tag{149}$$

where μ *is constant.*

Proof:

Let $\boldsymbol{\Phi}(t)$ be a *fundamental matrix* for the system such that

$$\dot{\boldsymbol{\Phi}}(t) = \mathbf{P}(t)\boldsymbol{\Phi}(t)\,. \tag{150}$$

this results in

$$\dot{\boldsymbol{\Phi}}(t+T) = \mathbf{P}(t)\boldsymbol{\Phi}(t+T)\,, \tag{151}$$

because of the periodicity in $\mathbf{P}$. From Theorem 3.10 we have that $\det(\boldsymbol{\Phi}(t+T)) \neq 0$. Therefore this is also a fundamental matrix which can be expressed by the first matrix, thus

$$\boldsymbol{\Phi}(t+T) = \boldsymbol{\Phi}(t)\mathbf{E} \tag{152}$$

where $\mathbf{E}$ is a constant matrix which has a determinant different from zero because $\det(\boldsymbol{\Phi}(t+T)) = \det(\boldsymbol{\Phi}(t))\,\det(\mathbf{E})$ (Theorem 3.10).

Let μ be the eigenvalues to $\mathbf{E}$, so that $\det(\mathbf{E} - \mu\mathbf{I}) = 0$. Furthermore let $\mathbf{s}$ be the eigenvector corresponding to the eigenvalues μ such that $(\mathbf{E} - \mu\mathbf{I})\mathbf{s} = \mathbf{0}$.

We consider the solution $\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{s} \stackrel{def}{=} \boldsymbol{\chi}(t)$. We obtain

$$\begin{aligned} \boldsymbol{\chi}(t+T) &= \boldsymbol{\Phi}(t+T)\mathbf{s} \\ &= \boldsymbol{\Phi}(t)\mathbf{E}\mathbf{s} = \boldsymbol{\Phi}(t)\mu\mathbf{s} \\ &= \mu\,\boldsymbol{\chi}(t)\,. \end{aligned}$$

Q.E.D

The eigenvalues to the matrix $\mathbf{E}$ we call characteristic numbers for Eq. (148). The interesting feature here is that there exist characteristic numbers with values that result in periodic solutions.

Theorem 3.18 (The constants μ) *The constants μ in Theorem 3.17 is independent of the choice of the fundamental solution $\mathbf{\Phi}$.*

Proof:

Let $\mathbf{\Phi}$ and $\mathbf{\Phi}^*$ be two fundamental matrices connected by the constant matrix $\mathbf{C}$ such that $\mathbf{\Phi}^* = \mathbf{\Phi}\mathbf{C}$. Furthermore let T be the minimum period for $\mathbf{P}(t)$. Then we find

$$\begin{aligned}\mathbf{\Phi}^*(t+T) &= \mathbf{\Phi}(t+T)\mathbf{C} \\ &= \mathbf{\Phi}(t)\mathbf{E}\mathbf{C} \\ &= \mathbf{\Phi}^*(t)\mathbf{C}^{-1}\mathbf{E}\mathbf{C} \\ &= \mathbf{\Phi}^*(t)\mathbf{D}\end{aligned}$$

here we notice that the matrix introduced as $\mathbf{D}$ and the matrix $\mathbf{E}$ are similar matrices which means they have the same eigenvalues.

Q.E.D.

Notice that it is therfore relevant to refer to the characteristic numbers, μ, as something unique, something that is independent on how the solutions are represented. Furthermore we have that if $\mathbf{\Phi}$ is a real matrix, then so is $\mathbf{E}$, such that the characteristic equation for the numbers (eigenvalues) μ has real coefficients. Therefore to every complex characteristic number μ there corresponds a characteristic number which is the complex conjugate $\bar{\mu}$.

Definition 3.6 (Characteristic exponent) *Let μ be the characteristic number corresponding to Eq. (148). Let ρ be defined by* $\exp(\rho T) \stackrel{def}{=} \mu$*, her ρ is called the characteristic exponent for the system.*

We notice that ρ is defined such that one may add an arbitrary multiple of $2\pi i/T$. Usually μ is fixed by the condition $-\pi < \Im(\rho T) < \pi$, or by $\rho \stackrel{def}{=} \frac{1}{T}\log(\mu)$, where the principle value of the logarithm is used.

Theorem 3.19 (Floquet representation) *Suppose that $\mathbf{E}$ in the proof for Theorem 3.17 has n different (distinct) eigenvalues, μ_i , $\quad i = 1, 2, \ldots, n$. Then Eq. (148) has n linearly independent normal solutions of the form*

$$\mathbf{x}_i = \mathbf{p}_i(t)e^{\rho_i t} \tag{153}$$

where ρ_i are the characteristic exponents corresponding to μ_i, and $\mathbf{p}_i$. The functions $\mathbf{p}_i$ are periodic functions with period T.

Proof:

To every μ_i there corresponds a solution $\mathbf{x}_i$ which is a solution to Eq. (149): $\mathbf{x}_i(t+T) = \mu_i\mathbf{x}_i(t) = \mathbf{x}_i(t)e^{\rho_i T}$. Therefore for arbitrary t we have that

$$\mathbf{x}_i(t+T)e^{-\rho_i(t+T)} = \mathbf{x}_i(t)e^{-\rho_i t}\,. \tag{154}$$

With $\mathbf{p}_i(t) \stackrel{def}{=} \mathbf{x}_i(t)e^{-\rho_i t}$ one sees that $\mathbf{p}_i(t)$ has the period T. Q.E.D.

3.8.1 Stability of a periodic system

From Eq. (148), where $\mathbf{\Phi}(t)$ is a fundamental matrix, we obtain that $\mathbf{\Phi}(t+N\,T)$ is also a fundamental matrix, since $\mathbf{P}(t+N\,T) = \mathbf{P}(t)$. Therefore there exists a constant matrix $\mathbf{C}$ such that

$$\mathbf{\Phi}(t+N\,T) = \mathbf{\Phi}(t)\mathbf{C}\,. \tag{155}$$

By choosing the fundamental matrix such that $\mathbf{\Phi}(0) = \mathbf{I}$ (the unit matrix), we easily find by putting $t = 0$ in Eq. (155) that $\mathbf{C} = \mathbf{\Phi}(N\,T)$. Furthermore by taking $t = T$ in Eq. (155), we obtain

$$\mathbf{\Phi}((N+1)\,T) = \mathbf{\Phi}(T)\mathbf{\Phi}(N\,T) \tag{156}$$

by induction we then obtain

$$\mathbf{\Phi}(N\,T) = \mathbf{\Phi}^N(T) = \mathbf{B}^N \quad \text{where} \quad \mathbf{B} \stackrel{def}{=} \mathbf{\Phi}(T)\,, \tag{157}$$

such that

$$\mathbf{\Phi}(t+N\,T) = \mathbf{\Phi}(t)\mathbf{B}^N\,. \tag{158}$$

Now we may always write an arbitrary t as $t = p + N\,T$ where $p \in (0,T)$. This way we obtain the result in Eq (158) with $0 < t < T$.

Theorem 3.20 *If $\mathbf{\Phi}(t)$ is limited for $t \in (0,T)$, then $\mathbf{\Phi}(t)$ will be limited for all t if and only if $\mathbf{B}^N$ is limited when $N \to \infty$.*

Proof:

From Eq. (158) with $t = p \in (0,T)$ one finds

$$\|\mathbf{\Phi}(t)\| = \|\mathbf{\Phi}(p+N\,T)\| \leq \|\mathbf{\Phi}(p)\|\|\mathbf{B}^N\|\,,$$

which shows that $\|\mathbf{\Phi}(t)\|$ is limited if $\|\mathbf{B}^N\|$ is limited.

Furthermore, also from Eq. (158), since $\det(\mathbf{\Phi}(p)) \neq 0$ (Theorem 3.10), we obtain

$$\mathbf{\Phi}(p)^{-1}\mathbf{\Phi}(p+N\,T) = \mathbf{B}^N\,.$$

Then since $\|\mathbf{\Phi}(p)^{-1}\|$ is limited we find

$$\|\mathbf{B}^N\| \to \infty \quad \Rightarrow \quad \|\mathbf{\Phi}(p + N\,T)\| \to \infty\,.$$

Q.E.D.

Definition 3.7 (Spectral radius) *Let* $\mathbf{B}$ *be a matrix having eigenvalues* λ_i $(i = 1,\, 2,\, \ldots)$. *The spectral radius*

$$r(\mathbf{B}) \stackrel{def}{=} \max_i |\lambda_i|$$

.

Theorem 3.21 (Asymptotic stability) *Eq. (148) has asymptotically stable solutions if and only if* $\mathbf{B}$ *given in Eq. (158) has a spectral radius* $r(\mathbf{B}) < 1$. *If* $r(\mathbf{B}) = 1$, *the solution is still stable if there is not coinciding eigenvalues.*

Proof: If we choose a representation where $\mathbf{B}$ is diagonal, that is the diagonal elements are the eigenvalues to $\mathbf{B}$, $\lambda_i\,,\ i = 1,\, 2,\, \ldots\,, n$, then $\mathbf{B}^N$ will be diagonal with the diagonal elements λ_i^N. The result then follows from Theorem 3.20.

Q.E.D.

The problem of finding $\mathbf{B}$, and thereby the spectral radius analytically can often be difficult, and one needs to fall back on numerical methods. But in the example, 3.3, it is possible to find an analytic expression.

Example 3.3 (Hill's equation) *Hill's equation can be written as*

$$\ddot{x} + F(t)\,x = 0 \quad \text{where} \quad F(T + t) = F(t)\,. \tag{159}$$

In matrix form it corresponds to Eq. (148), and is

$$\mathbf{P}(t) = \begin{pmatrix} 0 & 1 \\ -F(t) & 0 \end{pmatrix}. \tag{160}$$

We notice that the trace $tr(\mathbf{P}(t)) = 0$ *(or* $\nabla \cdot \mathbf{X} = 0$*) which shows that this system is conservative, thus the area is conserved during motion in the phase plane. The mapping represented by Eq. (158) has a Jacobian determinant equal 1 or* $\det(\mathbf{B})$*=1. The eigenvalues of* $\mathbf{B}$ *satisfies the characteristic equation*

$$\lambda^2 - \mathrm{tr}(\mathbf{B})\lambda + \det(\mathbf{B}) = \lambda^2 - \mathrm{tr}(\mathbf{B})\lambda + 1 = 0\,, \tag{161}$$

This results in three cases:

(i) $|\mathrm{tr}(\mathbf{B})| > 2$*: Eq. (161) has two real roots, where one root is always greater than* 1, *since* $\lambda_1\,\lambda_2 = 1$. *Therefore there exists a solution to Eq. (159), where the spectral radius* $r(\mathbf{B}) > 1$, *which results in an unstable solution.*

(ii) $|\mathrm{tr}(\mathbf{B})| < 2$*: Eq. (161) has complex roots (complex conjugate) which are located on the circle having radius 1,* $r(\mathbf{B}) = 1$, *thus the solutions to (159) in this case are stable.*

(iii) $|\mathrm{tr}(\mathbf{B})| = 2$ *This is a limiting case when Eq. (161) has a* T*-periodic (or* $2T$*-periodic solution).*

When $\mathrm{tr}(\mathbf{B}) = 2$ *one has coinciding roots* $\lambda_1 = \lambda_2 = 1$*. If we choose the corresponding eigenvector as the initial vector in Eq. (158), we find* $\mathbf{x}(T) = \mathbf{x}(0)$*, that is we have a* T*-periodic solution.*

When $\mathrm{tr}(\mathbf{B}) = -2$ *one has coinciding roots* $\lambda_1 = \lambda_2 = -1$*. If we choose the corresponding eigenvector as the initial vector in Eq. (158), we find* $\mathbf{x}(T) = -\mathbf{x}(0)$ *and* $\mathbf{x}(2T) = -\mathbf{x}(T) = \mathbf{x}(0)$*, thus we have a* $2T$*-periodic solution.*

Notice that in general one may write a solution as $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0$, where $\mathbf{x}_0$ is the initial vector. Then if the result in Eq. (158) is used, one can easily show the results under (iii) above.