The graphical iteration has two steps:

1. Go vertically to the graph $f(x)$.
2. Go horizontally to the graph $y = x$.

This is repeated to we reach the goal if the process converge.

Fig. 10 shows this process for $\lambda = \frac{1}{2}$, and the two *fixed points* are marked with stars.

From this figure it is self evident that if we start in an arbitrary point in the interval $(0, 1)$, then the iteration process will lead to the *fixed point* $x_0^* = \frac{1}{2}$. Notice that however close we start to the *fixed point* in $x = 0$, the process is going to diverge and finish at the *fixed point* $x_0^* = \frac{1}{2}$. The *fixed point* $x = 0$ with this property we call *unstable*.

On the other hand we have that if we start close enough to the other *fixed point* located in $x = \frac{1}{2}$ (this includes all points in $(0, 1)$), then we finish in this *fixed point* , which we for obvious reasons then call a *stable fixed point* . We also call such a *fixed point* for an *attractor* with period 1.

If we do not care about the transient behavior of the iteration process, but only focus on the regular behavior as an eventual end result, then the information about the stable *fixed point* , $x = y = \frac{1}{2}$, is what we look for as the finishing point in the iteration process. By this approach, we should notice that it does not matter where we start only that we are sufficiently close, in *the region of attraction*, regarding the stable *fixed point* . Thus the end result will be independent of starting point as long as we start inside *the region of attraction*. To determine exactly *the region of attraction* can be a difficult problem. But trial and error often quickly give the desired result. The attractor, (*fixed point*), satisfies Eq. (206), which, as stated above, is independent of the starting point x_0.

This condition is a key question in the universal behavior: If an *attractor* exists then the behavior is independent on which point the process starts at.

What makes $x = 0$ unstable, but $x = \frac{1}{2}$ stable?

It should not be to difficult to convince oneself that $x = 0$ is unstable because the *slope for* $f(x)$ $(f'(x))$ is larger than 1 in this point. In fact we have that if x^* is a *fixed point* for f and the derivative of f, evaluated in x^*, $|f'(x^*)|$ is less than 1, then x^*, is a stable *fixed point* . On the contrary if $|f'(x^*)| > 1$, than x^* is an unstable *fixed point* . It is also clear that only stable *fixed points* can be attractors for the iteration process which starts in an arbitrary point.

The next question is: For which values of λ are the *fixed points* attractive? This is illustrated graphical in Fig. 11.

From Eq. (205) we find

$$f'(x) = 4\lambda(1 - 2x) \tag{207}$$

such that

$$f'(0) = 4\lambda\,, \tag{208}$$

and with $x_0^* = 1 - \frac{1}{4\lambda}$, we find

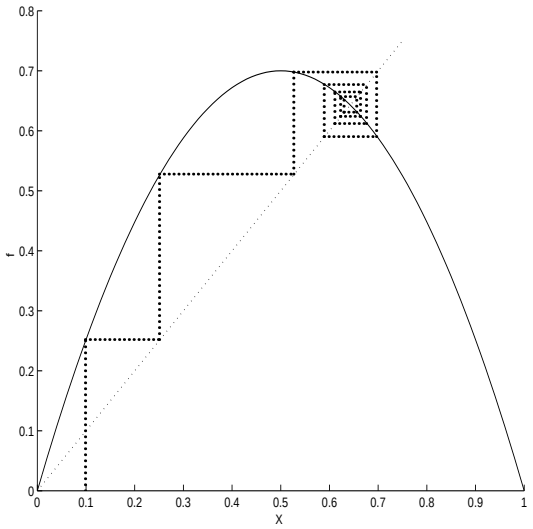


Figure 11: *Feigenbaum plot for the logistic equation. The iterated of* $x_0 = 0.1$ *for* $\lambda = 0.7$ *(with 10 iterations).*

$$f'(x_0^*) = 2 - 4\lambda\,. \tag{209}$$

We then see that for $0 < \lambda < \frac{1}{4}$ it is only $x^* = 0$ which is a stable *fixed point* . For $\lambda = \frac{1}{4}$ we have $x_0^* = 0$ and $f'(x_0^*) = 1$. For $\frac{1}{4} < \lambda < \frac{3}{4}$ we have $x^* = 0$ unstable and x_0^* stable. But for $\lambda = \frac{3}{4}$ we have $f'(x_0^*) = -1$ and also x_0^* has become unstable. From this result we see that in the region $0 < \lambda < \frac{3}{4}$ we know the behavior of the system with regard to iterations.

What happens to the system in the region $\frac{3}{4} < \lambda < 1$, when there is no stable or attractive *fixed points* ? We shall see that when λ increases and become somewhat larger than $\frac{3}{4}$, then the system will start the scenario of period doubling. This means that in stead of a stable cycle of period 1, we find that corresponding to the fixed point, the system enters into a stable cycle of period 2. This means that this stable cycle now contain *two* points, and these two points are *fixed points* for f^2. Since stability of these points are determined by the slop for f^2, we now focus on the behavior of f^2 around these two *fixed points* .

In order to understand Fig. 12, it is important to notice that since f is symmetric around the maximum point $x = \frac{1}{2}$, so is also f^2 symmetric around this point. Furthermore f^2 must have a *fixed point* every time f has fixed point because the second iterated of a *fixed point* still continues to be a *fixed point* . The most important property that determines the evolution of period doubling for f for increasing λ, is the condition related to the ratio of the slopes for f^2 and f. This ratio is determined from taking derivatives using the chain rule. We obtain

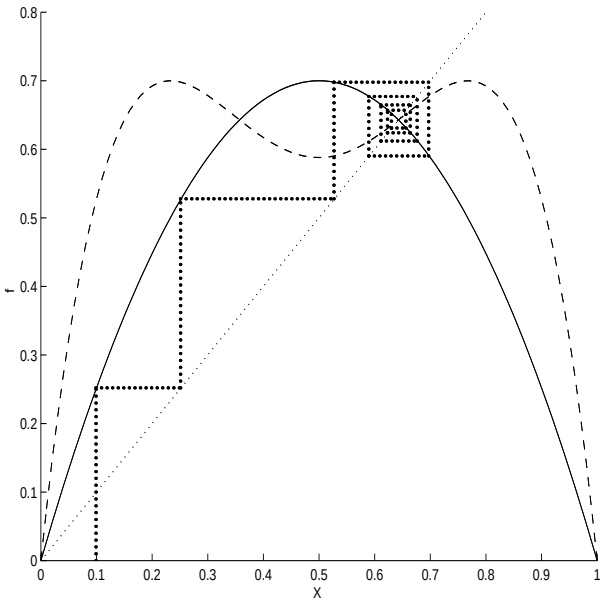


Figure 12: *Feigenbaum plot for the logistic equation. The iterated of* $x_0 = 0.1$ *for* $\lambda = 0.7$ *where the curve with double hump is the second iterated,* $f^2(x)$ *(with 10 iterations).*

$$x_2 \stackrel{def}{=} f^2(x_0)\,, \tag{210}$$

where

$$x_1 = f(x_0)\,, \quad x_2 = f(x_1)\,. \tag{211}$$

One can easily verify by use of the chain rule that

$$f^{2'}(x_0) = f'(x_0)f'(x_1)\,, \tag{212}$$

and

$$f^{n'}(x_0) = f'(x_0)f'(x_1)\cdots f'(x_{n-1})\,, \tag{213}$$

where the notation $f^{n'}$ means the derivative of the n^{th}- iterated of f. This elementary result determines period doubling. If we start in a *fixed point* for f and use Eq. (212), where we set $x_0 = x^*$, such that $x_2 = x_1 = x^*$, then we obtain

$$f^{2'}(x^*) = f'(x^*)f'(x^*) = [f'(x^*)]^2\,. \tag{214}$$

Since we for $\lambda = 0.7$ have that $|f'(x^*)| < 1$, it follows from Eq. (214) that

$$0 < f^{2\prime}(x^*) < 1\,. \tag{215}$$

We see also from Eq. (213) that if we start in the maximum point for f, $x_0 = \frac{1}{2}$, then we have

$$f^n(\frac{1}{2}) = 0\,, \tag{216}$$

for all n.

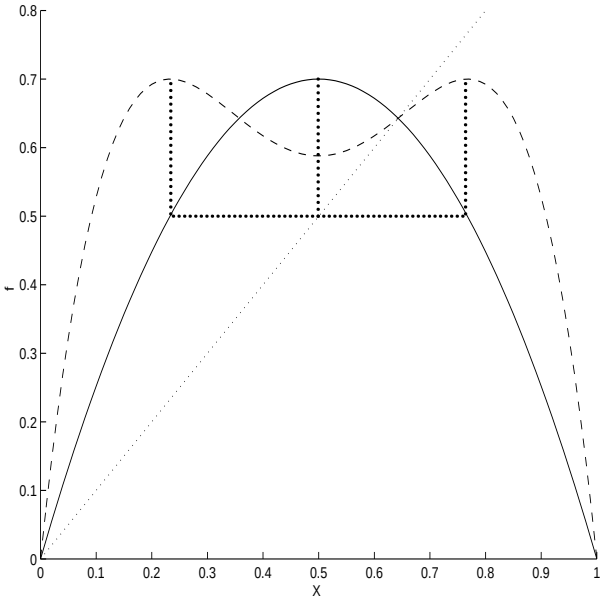


Figure 13: Here f and f^2 are displayed without iterations, but the splitting which occurs is because of the inverse of f is not single valued ($\lambda = 0.7$).

Especially we have that f^2 has an extremal point (minimum) in $x = \frac{1}{2}$, and from Eq. (212) we see that f^2 has an extremal point (maximum) in x_0, which under f will iterate to $x = \frac{1}{2}$, because we then have $x_1 = \frac{1}{2}$ and $f'(x_1) = 0$. These points, the inverse of $x = \frac{1}{2}$ can be found by moving vertically downward along $x = \frac{1}{2}$ until intersection with the line $y = x$, and then horizontally to $y = f(x)$. Since f has a maximum, there are two intersection points. These are connected to the *two* maximum points we have on the dashed curve which represents the graph of f^2. See Fig. 12 and 13.

The property of complex behavior is a consequence of the fact that , $f^{-1}(\frac{1}{2})$, is not single valued (in this case two values), as seen in Fig. 13.

A monotonic f, that is one that always increase, always has a simple behavior whether this behavior is computable or not (for example when f is nonlinear). Those functions f being interesting in our case, always folds (turns around) because of its nonlinearity. It is just this process of folding because of nonlinear behavior which is the very basis for our analysis.

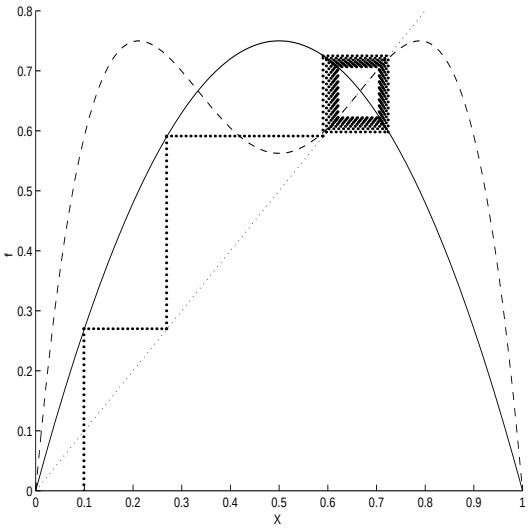


Figure 14: *Feigenbaum plot of the logistic equation. The iterate of* $x_0 = 0.1$ *for* $\lambda = 0.75$ *(with 20 iterations).*

We return to Fig. 12 and investigate what happens when $\lambda \to 0.75$, it is then quite apparent from Fig. 14 that we have extremely slow convergence.

It is clear from Fig. 12 and Fig. 15 (a) and (b), where λ is respectively 0.7 and 0.785, that is: somewhat less and somewhat larger than $\frac{3}{4}$; that the iteration process changes character from converging towards *one point* to "converge " towards *two points*. Whereas one in Fig. 14 we have the marginal case which converges very slowly. In Fig. 15 (a) we notice that the iteration process converges alternatively towards a x_1^* and a x_2^* which are the intersection points in between the straight line $y = x$ and $y = f^2(x)$, such that we have found a solution which function so that $f(x_1^*) \to x_2^*$ and $f(x_2^*) \to x_1^*$. Such a pair of points we call a *2-periodic* solution, in spite of the endless loop we arrive at in the iteration process in the external square in . Fig. 15 (a). This means that the sequence

$$x_0,\, x_1,\, x_2,\, x_3,\, \ldots\,, \tag{217}$$

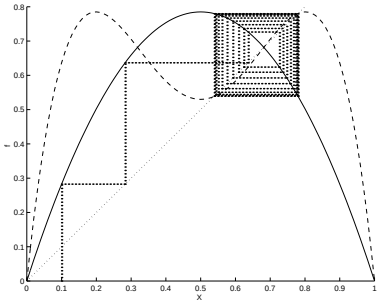
approaches closer and closer to

$$x_1^*,\, x_2^*,\, x_1^*,\, x_2^*,\, \ldots\,, \tag{218}$$

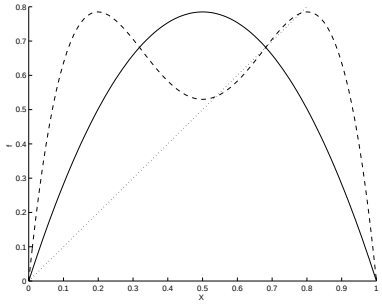
therfore this is a stable 2-cycle or *attractor* with period 2.

Also notice that *the slop* for $f^2(x)$ is greater than 1, in the region around the fixed point for $f(x)$ therfore this is an unstable fixed point for $f^2(x)$ which are common with $f(x)$ and $f^2(x)$. Fig. 15 (b) gives a more clear picture of this case which is the same as Fig. 15 (a), but without the iteration process.

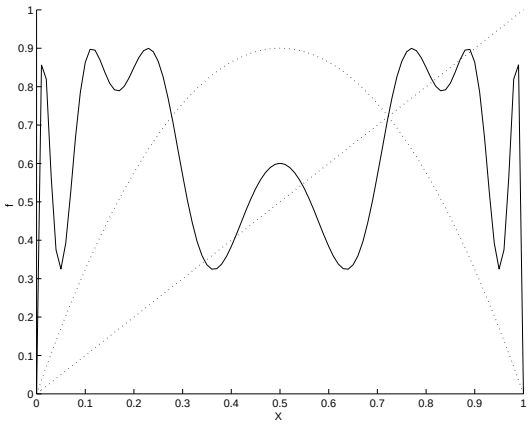
We conclude that we have observed the first period doubling (with reference to Eq. (205)) with increasing parameter λ.)



(a) Med iterations



(b) Without iterations



(c) Plot of $y = f^4(x)$, $y = f(x)$ and $y = x$.

Figure 15: *Feigenbaum plot for the logistic equation. The iterate of* $x_0 = 0.1$ *for* $\lambda = 0.785$ *where the curve having a double hump is the second iterate,* $f^2(x)$ *(with 20 iterations) , (a) with iterations and (b) without. The figure (b) shows the fixed points for* $f(x)$ *and* $f^2(x)$ *as intersection with the dotted line* $y = x$*. In (b) the curves for* $y = f(x)$ *and* $y = x$ *are dashed and* $\lambda = 0.9$*. The figure shows the fixed points for* $f(x)$ *and* $f^4(x)$ *as the intersection with the dotted line* $y = x$*.*

Another feature which is important to pay attention to is that f^2 has the same slope in x_1^* and x_2^*. This is a consequence of Eq. (212), since $x_0 = x_1^*$ and $x_1 = x_2^*$ and vice versa, thus the product of the numbers that is a measure for the slopes becomes equal. In general if x_1^*, x_2^*, $x_3^*, \ldots, x_n^*$, is a n-cycle such that

$$x_{r+1}^* = f(x_r^*)\,, \qquad r = 1,\, 2,\, \ldots\,, n-1\,, \tag{219}$$

and

$$x_1^* = f(x_n^*)\,, \tag{220}$$

then every x_r^* is a fixed point for f^n with identical slops:

$$x_r^* = f^n(x_r^*)\,, \qquad r = 1,\, 2,\, \ldots\,,\, n\,, \tag{221}$$

and

$$f^{n\prime}(x_r^*) = f'(x_1^*)\ldots f'(x_n^*)\,. \tag{222}$$

This implies in turn period doubling *ad infinitum*.

By further increasing λ the minimum in $x = \frac{1}{2}$ will decrease since the slope for f^2 corresponding to the fixed point for f will increase. In this process one reaches the λ-value, λ_1^* which is such that $x = x^* = \frac{1}{2}$ becomes a fixed point for f^2. Simultaneously the maximum point to the right will also become a fixed point for f^2.

We now have that the top point for $f(x)$ is located on this path, and from Eq. (222) we have $f^{2\prime}(x^*) = 0$. When this happens, we say that the path is *super stable*. Super stable paths exist for those λ-values which are listed in the table 1. A MAPLE-program for computing the first λ^*-values is presented in Appendix B.

Furthermore from the formula for δ_n in Eq. (199) we find by using the first 5 λ^*-values in the table 1, the following values:

λ^*	=	0.5000000	(super stable period -1 bane)
λ^*	=	0.8090170	(super stable period -2 bane)
λ^*	=	0.8746404	(super stable period -4 bane)
λ^*	=	0.8886602	(super stable period -8 bane)
λ^*	=	0.8916667	(super stable period -16 bane)
λ^*	=	0.8923110	(super stable period -32 bane)
λ^*	=	0.8924482	(super stable period -64 bane)
λ^*	=	0.8924782	(super stable period -128 bane)
λ^*	=	0.8924847	(super stable period -256 bane)

Table 1: $\lambda-$ values for super stable paths.

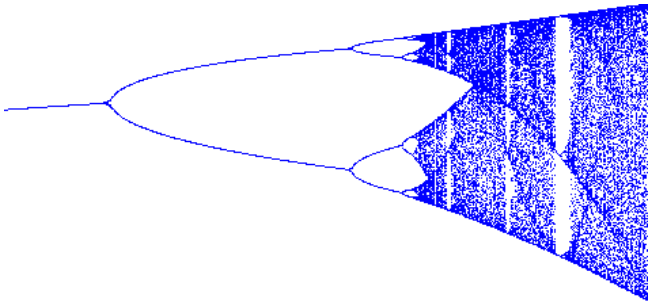
$$\begin{aligned}
\delta_1 &= \frac{\lambda_2^* - \lambda_1^*}{\lambda_3^* - \lambda_2^*} = 4.68075 \\
\delta_2 &= \frac{\lambda_3^* - \lambda_2^*}{\lambda_4^* - \lambda_3^*} = 4.66316 \\
\delta_3 &= \frac{\lambda_4^* - \lambda_3^*}{\lambda_5^* - \lambda_4^*} = 4.66666 \\
\delta &= 4.6692016... \qquad \text{correct value}\,.
\end{aligned}$$

In Fig. 15 (c) we see what happens further on when λ increases and we get four fixed points for f^4 in addition to the fixed points we already have for f.

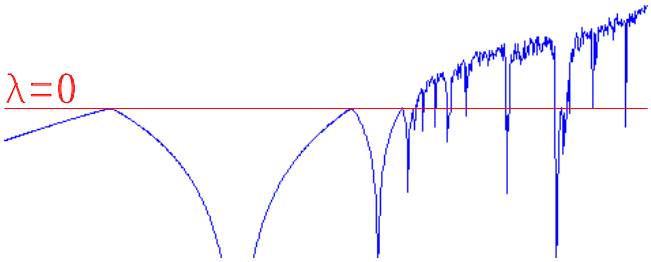
When λ continues to increase we get the bifurcation-diagram which is presented in Fig.16 (a) with a corresponding diagram for the Liapunov-exponent for this problem in Fig. 16 (b),where the axes in these two diagrams should coincide.

Notice especially that the Liapunov-exponent approaches zero from the negative side in the first two bifurcation points and then crosses over and becomes positive (unstable).

Qualitative what happens when λ increases, is that the equilibrium points (the fixed points) for higher and higher iterate come into play and produces periodic solutions with period length $2^n\,T$, where f^{2n} has this period.



(a) *Plot of bifurcation in the logistic mapping (λ is varying along the horizontal axis)*

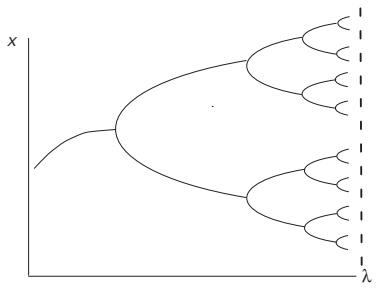


(b) *Plot, of Lyapunov-exponent in the logistic mapping (λ is varying along the horizontal axis)*

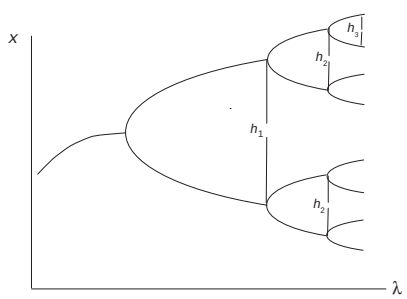
Figure 16: *Plot (a) of a bifurcation process and the corresponding plot (b) displaying the Lyapunov-exponent*

4.2 Universal limit for high iteration

In Fig. 15 (c) we see, by studying the local maximum for f^4 in centrum of the figure ($x = \frac{1}{2}$, $y = \frac{1}{2}$), we see that looked at locally, we are back to the situation from which we started. Continuing this process we expect a similar behavior to what we already have observed. Further more the scale which f^2 determines for f^4 is fully determined by the functional composition, so that seen locally the curves corresponding to f^{2n} and f^{2n+1} converge such that the scale reduction from one level to the next will *converge to a certain constant*.



(a) *A sketch of principle for period doubling for the logistic mapping*



(b) *A sketch of principle for period doubling for the logistic mapping with distance in x-value for each step.*

Figure 17: *Plot (a) of a general period doubling process. (b) same process with jumps in x-value for different levels in the evolution.*

Let d_n denote the distance from $x = \frac{1}{2}$ to the nearest element of the attractor cycle having a period 2^n in the 2^n-cycle for $\lambda = \lambda_n$. Then there exists a positive number α_n which scales this distance down to the 2^{n+1}-cycle for $\lambda = \lambda_{n+1}$ given by

$$\alpha_n \stackrel{def}{=} -\frac{d_n}{d_{n+1}}\,. \tag{223}$$

One can also show that this rescaling can be done by looking at the ratio $\frac{h_n}{h_{n+1}}$ like it is done schematic in Fig. 17 (a) and (b).

Since this rescaling is determined from only one functional composition, then there exists some function that will reproduce itself, scaled by the scale factor $-\alpha$. This function, g, has a quadratic maximum around $x = \frac{1}{2}$, and can be scaled to be 1 at $x = \frac{1}{2}$. By a coordinate transformation we get $x = \frac{1}{2} \to x = 0$, in fact it can be shown that g satisfies the functional equation

$$-\alpha\, g(g(\frac{x}{\alpha})) = g(x)\,. \tag{224}$$

We are not giving any detailed proof for this statment, however by substituting $g(0) = 1$, we find

$$g(1) = -\frac{1}{\alpha}\,. \tag{225}$$

We have that Eq. (224) is determined as equation and has maximum equal 1 in $x = 0$.

There exists a unique solution to Eq. (224) which determines the universal constant α,

$$\alpha = 2.502907875\ldots\,. \tag{226}$$

When α is known, then from Eq. (223) we can predict a unique scaling which connects all iterates in an arbitrary system which was built by the phenomenon of *period doubling*.

In Appendix C we have presented a MAPLE-program which find approximate values for α.

The "derivation" of the equation for g can be done rigorously as has been done for example by Feigenbaum [8], but this becomes too much material to cover in our context.

4.3 Bifurcations

Bifurcation processes is a well known phenomenon from certain kinds of non linear differential equations which contain a parameter that can vary, *the bifurcation parameter*. Then it can happen that a stable equilibrium, at a critical point becomes unstable simultaneously as two new equilibrium points emerge as shown in Fig. 18 (b).

4.3.1 Bifork bifurcation

A simple example of an equation that gives a "bifork bifurcation" is

$$\dot{x} = y\,, \qquad \dot{y} = \mu\, x - x^3 - y\,. \tag{227}$$

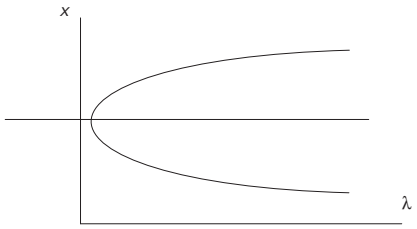
Here we have for $\mu < 0$, that the origin is a stable equilibrium point: A stable node for $-\frac{1}{4} < \mu < 0$, and a spiral for $\mu < -\frac{1}{4}$. For $\mu > 0$ we have that the origin is an unstable equilibrium point (a saddle-point). The other equilibrium points can be found and are $x = \pm\sqrt{\mu}$, $y = 0$. These points are stable nodes for $\mu < \frac{1}{8}$ and stable spirals for $\mu > \frac{1}{8}$. Here μ is the bifurcation parameter. See Fig. 18 (b).

4.3.2 Hopf-bifurcation

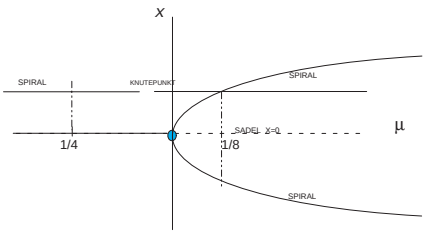
A standard example for another type of bifurcation is called Hopf-bifurcation. In this case we have a stable equilibrium point which becomes unstable and changes to a stable limit cycle. The following example illustrate this case

$$\dot{x} = \mu\, x + y - x(x^2 + y^2)\,, \tag{228}$$
$$\dot{y} = -x + \mu\, y - y(x^2 + y^2)\,, \tag{229}$$

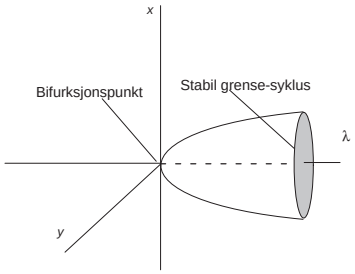
or written in polar coordinates



(a) *Plot of bifurcation, sketch for the general case*



(b) *Plot of bifurcation Eq. (227), where μ varies along the horizontal axis*



(c) Hopf-bifurcation

Figure 18: *Plot (a) of a general bifork bifurcation process. (b) refers to Eq. (227) and (c) is a sketch of a Hopf-bifurcation.*

$$\dot{r} = r(\mu - r^2)\,, \quad \dot{\theta} = -1\,.$$

For $\mu < 0$ the solution is a stable spiral which approaches $x = y = 0$ or $r = 0$. When μ increases and becomes greater than zero the stable spiral becomes unstable and approaches a stable limit cycle as illustrated in Fig. 18 (c).

4.4 Influence from small differences in initial conditions

It is typical for nonlinear problems being close to the chaotic regime that solutions on a long timescale can depend drastically on initial conditions. A classic example is the so called Duffing's equation with a forcing term, $r\cos(st)$. We write this equation as a system and find [11]

$$\begin{aligned} \dot{x}_1 &= -k\,x_1 - x_2 - x_2^3 - r\cos(s\,t)\,, \\ \dot{x}_2 &= x_1\,. \end{aligned}$$

With $k = 0.3$, $r = 0.5$ and $s = 1.2$ we obtain the following numerical solutions:

Fig. 19 shows that after approx. 1000 units on the horizontal axis or 10 to 11 large oscillations, there is little connection in between the solutions.

[11] This problem is solved using Matlab. The files are provided in Duffing.m and duff.m which are made available on the web.

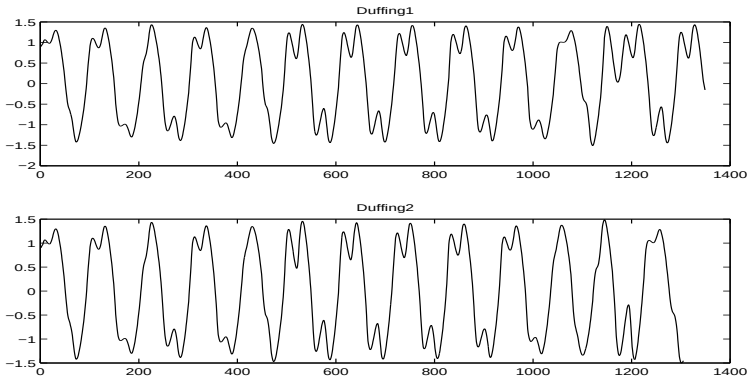


Figure 19: *Plot of the solution for Duffing's equation with initial conditions* $x_1(0) = 0.900000$ *and* $x_2(0) = 0.42$ *in the upper part of the figure and* $x_1(0) = 0.900001$ *and* $x_2(0) = 0.42$ *in the lower part of the figure.*

In conclusion we have that it may not be too meaningful to solve a nonlinear initial value problem for a long time horizon.

4.5 "Strange" attractors and fractals

The results we have found that evolve towards a chaotic behavior, lead us to another subject, fractals. We already have encountered a couple of cases with attractors in the plan which are distinctly different from what we normally consider to be curves. First of all, a curve in two dimensions have no area. However, when we look at the strange attractors we found in connection with Duffing's equation Fig. 9 and in connection with the logistic equation Fig. 16 (a), we notice that these structures look different from what we consider to be curves, and for special reasons we call these structures *fractals*. We shall present the concept of a fractal by considering a simple set due to Cantor, called the Cantor set. This set is constructed in the following way. We consider an interval for example $[0, 1]$, we divide this interval in three equal parts and remove the middle part. In the two remaining intervals we do the same procedure, divide each interval in three equal parts and in each case remove the middle one. This process can be continued n times. In the limit $n \rightarrow \infty$ we end up with a set defined as the Cantor-set. After n steps in the process there remain 2^n line segments which each has a length $(\frac{1}{3})^n$. The measure or total length of the Cantor-set is

$$1 - \sum_{k=1}^{\infty} 2^{k-1}(\frac{1}{3})^k = 0\,. \tag{230}$$

In order to measure such sets as the one above we need a new concept: One alternative is called the *capacity-dimension*.

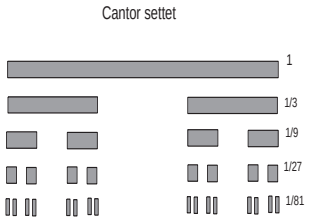


Figure 20: Graphic picture of the process of constructing a Cantor-set

4.6 Dimension

Every set in mathematics has a characteristic property: *Dimension*. This also applies to attractors in phase space or to a Poincaré-section in phase space. Natural numbers of dimensions are: 0 (a point), 1 (a line segment), 2 (an area, surface-segment), 3 (a volume) and so forth. Higher dimensions often are named as hyper surfaces or hyper volumes, even though such sets can not be visualized geometrically (we are confined to three dimensions or less regarding visualization, however this does not put any limitation on what is possible in a mathematical sense).

Lately it has become rather popular to study sets with dimensions not being a whole number. But in order to do so we need an operational definition, that is a definition that makes it possible in concrete cases to decide which dimension a given set has in the case where the dimension for obvious reasons is not a whole number. At the beginning of last century it felt important to understand and clarify this problem. However, as time has passed on complications and difficulties have been steadily increasing. Presently, probably tens of definitions exist regarding dimension. In other words we have a complicated situation to deal with.

In our context we have to make a shortcut and concentrate on a simple approach letting our evaluation of dimension rest on what is called *capacity dimension*.

4.6.1 Capacity-dimension

Let the actual set of points in the n-dimensional space, of which we want to determine the dimension, be given and limited, that is the distance between two arbitrary points in the set should have an upper limit. Then let ϵ be the side in a n-dimensional cube. Furthermore let $N(\epsilon)$ be the smallest number of such cubes that is needed to cover all the points in the actual set.

We define the *capacity-dimension* by letting $N(\epsilon) = k\,\epsilon^{-D}$ such that $\ln N(\epsilon) = \ln k + D\ln\epsilon^{-1}$ or $D = \frac{\ln N(\epsilon)}{\ln(\epsilon^{-1})} - \frac{\ln k}{\ln \epsilon^{-1}}$, where k is a constant of proportionality. This leads to the following formal definition of dimensionality D:

$$D \stackrel{def}{=} \lim_{\epsilon\to 0} \frac{\ln N(\epsilon)}{\ln \epsilon^{-1}}\,. \tag{231}$$

As a test let us use this definition to determine the dimensionality of a line segment of length a.

Subdivide the line segment in n equal parts (a/n) where n is arbitrary large. Set $\epsilon = 1/n$. We then need $n\,a$ of these line segments in order to cover a, such that $N(\epsilon) = n\,a$, and we find

$$\begin{aligned} D &\stackrel{def}{=} \lim_{\epsilon\to 0} \frac{\ln N(\epsilon)}{\ln \epsilon^{-1}} \\ &= \lim_{n\to\infty} \frac{\ln(a\,n)}{\ln n} \\ &= \lim_{n\to\infty} \frac{\ln(a)}{\ln n} + 1 = 1\,. \end{aligned}$$

We notice that the length of the line segment do not enter in the answer and thus does not influence on the dimension. It is also easy to see that the dimension of a surface element $a \times b$ and a volume $a \times b \times c$ have the dimensions 2 and 3 respectively.

Returning to the Cantor-set, we chose $\epsilon = 3^{-k}$ and obtain

$$D = \lim_{k\to\infty} \frac{\ln 2^k}{\ln 3^k} = \frac{\ln 2}{\ln 3} = 0,631\,, \tag{232}$$

thus we arrive at a dimension that is not a whole number. A set with this property we call a $fractal$.

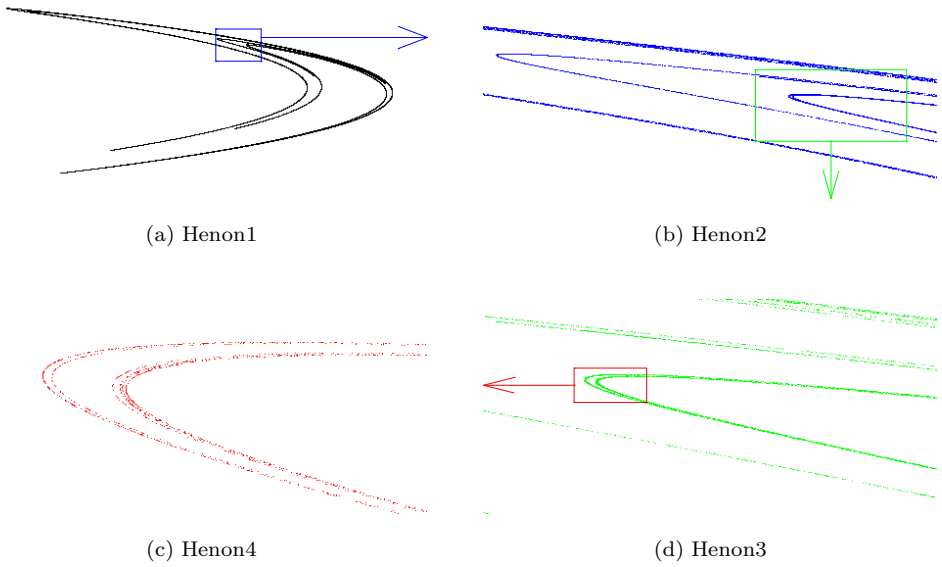
The examples we studied, i.e., Duffing's equation and the logistic equation also resulted in attractors that are fractals. See Fig. 9 and Fig. 17 respectively. Because of the complicated structure we obtain in both these cases we call these sets for *strange* attractors.

4.7 Henon attractor

An example that is often used to illustrate a *strange attractor* is the Henon attractor. The Henon attractor is the result from the mapping

$$x_{n+1} = y_n + 1 - a\,x_n^2 \tag{233}$$
$$y_{n+1} = b\,x_n \tag{234}$$

For $a = 1.4$ and $b = 0.3$ we obtain the numerical results which is shown in Fig. 21 and Fig. 22



(a) Henon1 (b) Henon2

(c) Henon4 (d) Henon3

Figure 21: *Henon-attractor with 3 enlargements* $1 \to 2 \to 3 \to 4$. *(The plot is produced by Dynamic Solver f32)*

From the enlargements we especially notice that we return to a structure similar to the one with which we started, but mirrored. If we had continued the enlargements three times more we expect to be back to the structure with which we started. This is typical for fractal structures, by enlargements they are self similar.

We also notice that in Fig. 22 we have a bifurcation pattern resembling the one we found in the logistic equation in Fig. 16 (a).

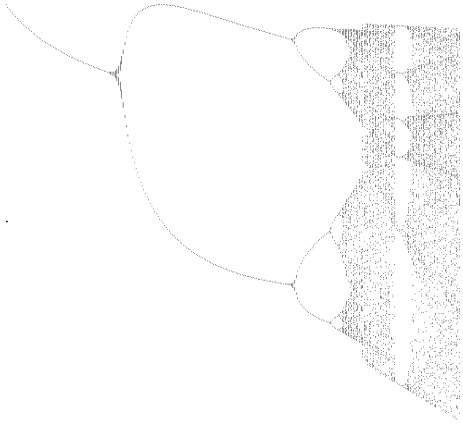
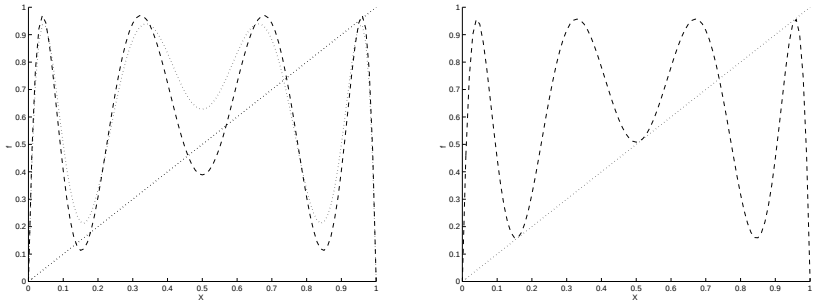


Figure 22: *Plot of bifurcation for the Henon attractor. (Plot produced by Dynamic Solver f32)*

In the book [9] there is given a table showing how the bifurcation parameter a varies with respect to the period doubling. This corresponds to our findings in the case of the logistic equation Feigenbaum's universal constant is a result from this system. It is very interesting to notice that even in a chaotic behavior there can exist a certain order, as the points make up an orderly pattern.

4.8 Period-3 Evolution

Until now we have concentrated on period-2 evolution. Could it also happen that fixed points for f^3 could have a similar evolution to that we found for f^2?



(a) periode-3 (b) periode-3

Figure 23: *Evolution of period-3. Fig. (a) displays two cases: The dotted curve has* $\lambda = 0,94$ *and the dashed curve has* $\lambda = 0,97$. *In Fig. (b) we have* $\lambda = 0,957$.

Fig. 23 is a graph of the 3-iterate, f^3, drawn for different values of the bifurcation parameter λ. We notice that when λ increase, we reach a critical λ-value where we get three new fixed points for f^3 which further on bifurcates in three new pares, that is six new fixed points of which three are stable and three unstable. Thus this is analogous to the period-2 evolution. Period-3 evolves in the same way to period-6, period-12, period-24 and so forth. This period doubling cascade has the same Feigenbaum-constant as before. This implies that increasing λ leads to chaos.

A well known result by Li & Yorke [11] expresses this as follows: " Period three means chaos". Li & Yorke studied the general mapping

$$x_{n+1} = F(x_n)\,, \tag{235}$$

where $F(x)$ is a continuous mapping on a closed interval for example $[0, 1]$ onto itself. Li & Yorke's theorem expresses that: If we have an F that maps a point a in b, that is $b = F(a)$ and b over in c, that is $c = F(b)$ and further c in d that is $d = F(c)$, such that: $d \leq a < b < c$, then the points in this interval will be mapped chaotic, that is in an arbitrary way. From this we see that if there exists a point a such that $d = a$, then we have a period-3 system and the conditions for the theorem of Li & Yorke are satisfied, which imply chaos. See Fig. 24

A property in such a system is that there exist points that have an arbitrary period. This follows from a theorem by Sharkovsky [12]:

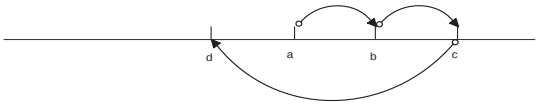


Figure 24: *Period - 3 senary.*

Theorem 4.1 (Sharkovsky) *Let T be an ordered set $\{3 < 5 < 7 < \cdots < 2 \times 3 < 2 \times 5 < 2 \times 7 \cdots < 2^2 \times 3 < 2^2 \times 5 < 2^2 \times 7 \cdots < 8 < 4 < 2 < 1\}$. Then let $f : I \to I = [0, 1]$, be a smooth mapping such that $f(0) = f(1) = 0$ which has one simple critical point. If $m < n$ relative to this ordering in the set T, and f has one periodic point with mark (that is the shortest period) m, then f has a periodic point with period n.*

4.9 Deterministic chaos

One problem area that has got attention the last twelve years or so is the chaotic behavior of deterministic systems. The fact that we do not need to interact with a system in an arbitrary way in order to produce chaos has important implications for philosophical reason. The Law of cause and effect, that often is taken as granted in a deterministic context, becomes a difficult subject when fully deterministic systems can provide chaotic behavior. But to explore this area and define *deterministic chaos* is a difficult subject that we can only give a brief account in our context.

If we look at a well known process, that of tossing a coin to determine head (H) or tail (T), then we have a process that tells us something about arbitrary behavior. Performing such a test we could for example obtain the following result

$$T,\ H,\ T,\ T,\ H,\ H,\ T,\ \ldots\ .$$

This describes a sequence of events. Such a sequence is called a *Bernoulli*-sequence. In our system we do not have any arbitrary input to the system, as we have by tossing a coin, still we can formulate the problem in such a way that it comes down to the question of comparing two sequences.

Definition 4.1 (Deterministic chaos) *A deterministic system is chaotic if we can find, for an arbitrary predetermined Bernulli-sequence, an initial point that dynamically will reproduce the same sequence of symbols, by moving through a selected part of phase space. If this happens we say that this initial point corresponds to the predetermined Bernulli-sequence.*

How can this be realized in practical terms? Let us look at the period-3 senary. We subdivide the interval for x, $[0,1]$, in bits that we name by using some symbol. If a path passes one of these bits, than we have realized the corresponding symbol. The claim is that there exists an initial value such that the corresponding path passes through all the prescribed bits. In this meaning the system is *chaotic*.

Li & Yorke also presented an alternative formulation. A consequence of the theorem of Sharkovsky 4.1, is that there exists an initial point that realizes an arbitrary predetermined period. This set is not even countable. Furthermore there is a set of non countable initial points that is not even asymptotically periodic, that is, thy do not approach a periodic behavior. This set of initial conditions we call S

Definition 4.2 (Asymptotic periodic) *Let q be a periodic point, then we call p asymptotic periodic if*

$$\lim_{n\to\infty} |F^n(p) - F^n(q)| \to 0\,. \tag{236}$$

This result tells us that there exist a non countable set, S, that does not even approach a periodic solution (and no points in S is periodic). Based on this Li & Yorke [11] has an alternative definition for chaos.

Definition 4.3 (Chaos, Li & Yorke) *For every p, $q \in S$ $(p \neq q)$, we have:*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0\,, \tag{237}$$

and

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0\,. \tag{238}$$

The interpretation of this is somewhat complicated and rests on the fact that every infinite sequence of points, $\{x_n\}$, in a closed set (compact) has at least one subsequence $\{x_{n_j}\} \in \{x_n\}$ which has a limit point (the Bolzano-Weierstrass theorem): Thus there exists a $l_{\{n_j\}}$, such that $\lim_{n_j\to\infty} x_{n_j} \stackrel{def}{=} l_{\{n_j\}}$, and this is dependent on the choice of sub sequense. From the definition of lim sup in Eq. (237), we have

$$\limsup_{n\to\infty} = \max_{\{n_j\}} l_{\{n_j\}} = l_{max}\,. \tag{239}$$

Correspondingly from Eq. (238), we have

$$\liminf_{n\to\infty} = \min_{\{n_j\}} l_{\{n_j\}} = l_{min}\,, \tag{240}$$

that is a minimum value obtained by looking at all possible subsequences. When this is used relative to Eq. (238) we obtain a sequence of values, $|F^n(p)-F^n(q)|$, which is obtained by arbitrary choice of initial points, $(p,q)\in S$, and this has a subsequence of mappings such that

$$\lim_{n_j\to\infty} |F^{n_j}(p) - |F^{n_j}(q)| = 0\,. \tag{241}$$

In other words, we can find mappings being located arbitrarily close to each other when $n_j\to\infty$, and on the other hand we can find sequences of mappings such that

$$\lim_{n_j\to\infty} F^{n_j}(p) = p_0\,, \qquad \lim_{n_j\to\infty} F^{n_j}(q) = q_0\,, \quad \text{and} \quad |p_0-q_0| = l_{max} > 0, \tag{242}$$

Eqs. (237) and (238) expresses that the mappings, F^n behave in an arbitrary manner (stochastic) in the limit $n\to\infty$, in the meaning that they remain arbitrarily close for certain steps n_j at the same time they have a finite distance (l_{max}) for other subset of such a mapping.

This is an interesting way to characterize a *chaotic* mapping. However it does not give any quantitative measure for *chaotic* behavior. This subject is, however, outside the frame for this course.

Let us step back and look at the logistic mapping. We remember there existed a critical value λ_∞ such that this value of λ made a distinction between periodic and aperiodic paths. For $\lambda > \lambda_\infty$ the path is characterized by being aperiodic and they never approach periodic paths, in addition such paths were very sensitive to initial values. In other words we have that for $\lambda > \lambda_\infty$ we are in a chaotic regime.

4.10 Practical use

It may now be appropriate to ask the question of practical use of all this theoretical discussion or whether it is only a game for especially interested mathematicians. By practical use we mean in this context whether this can be used to shed light on some practical phenomenon and model the behavior of such phenomenon. Research has revealed a number of cases where it is apparent that the process can be well explained by the period doubling cascade. Perhaps the most well known phenomenon in this respect is what is called Rayleigh-Bénard instabilities.

In a fluid heated from below, we get an unstable situation with a heavier fluid resting on the top of a lighter fluid acted upon by the force of gravity. This is because the effect of heating makes the fluid expand and therefore its specific weight decreases. In this case the bifurcation parameter is the temperature gradient. By increasing the temperature gradient the system becomes more and more unstable, this results in an orderly nonlinear motion in forms of "rolls". Further heating leads to "kinking rolls". Accurate observations reveal that this happen in correspondence with the period doubling scenario we have studied. A very interesting aspect with this phenomenon is that an inherently three dimensional system, shows the feature of one dimensional universality. See Fig. 25.

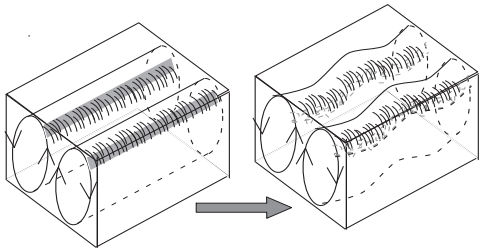


Figure 25: *Sketch of an experimental setup for Rayleigh- Bénard instability, in the nonlinear phase*

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Appendices

A Existence and uniqueness of solution of first order ordinary differential equations (ODE)

Given the differential equation and the initial condition

$$y'(x) = F(x, y(x)), \quad y(x_0) = y_0, \tag{243}$$

where a and y_0 are given values. We are going to formulate a theorem for existence and uniqueness for this problem. But first we consider a special method called *Successive approximations or Picard iterations*.

A.1 Successive approximations or Picard iterations

We are going to look for a solution of Eq. (243) by the following algorithm. We rewrite Eq. (243) as an integral equation

$$y(x) = y_0 + \int_a^x F(t, y(t)dt\,. \tag{244}$$

Notice that this formulation is equivalent to the inital value problem Eq. (243). Show this! We define the approximate solutions y_0, y_1, ... y_n:

$$\begin{aligned} y_1(x) &= y_0 + \int_a^x F(t, y_0(t))dt\,, \\ y_2(x) &= y_0 + \int_a^x F(t, y_1(t))dt\,, \\ &\vdots \\ y_n(x) &= y_0 + \int_a^x F(t, y_{n-1}(t))dt\,. \end{aligned}$$

and find the"solution" as

$$y(x) = \lim_{n\to\infty} y_n(x)\,. \tag{245}$$

In order to illustrate the method we look at a simple example:

$$y'(x) = k\,y(x), \quad x_0 = 0,\ y(0) = 1\,. \tag{246}$$

We obtain

$$
\begin{aligned}
y_1(x) &= 1 + \int_a^x k dt = 1 + kx\,, \\
y_2(x) &= 1 + \int_a^x k(1+kt) dt = 1 + kx + \frac{1}{2}(kx)^2\,, \\
y_3(x) &= 1 + \int_a^x k(1+kt+\frac{1}{2}(kt)^2) dt = 1 + kx + \frac{1}{2}(kx)^2 + \frac{1}{2\cdot 3}(kx)^3\,, \\
&\vdots \\
y_n(x) &= 1 + kx + \frac{1}{2}(kx)^2 + \frac{1}{2\cdot 3}(kx)^3 + \cdots \frac{1}{n!}(kx)^n\,,
\end{aligned}
$$

and further on

$$
y(x) = \lim_{n\to\infty} y_n(x) = e^{kx}\,, \tag{247}
$$

which we recognize as the exact solution of the problem Eq. (247). By the aid of successive approximations we were in this case able to recover the exact solution, expressed by an elementary function. But it is the exception that this happens. In the general case we find the solution in terms of a series, and if we are lucky, an expression for the general term in the series. But finding the sum of such a series in terms of a finite number of elementary functions is normally not possible. In this connection it is instructive to bring forth the simple differential equation, $y'' = xy$, Airy's equation where the solution can not be expressed in terms of elementary functions. An important feature with this method is that it can be used to find approximate solutions to problems that are difficult to solve in other ways. But some questions remain:

1. When does the process converge?
2. When does the process converge towards a solution of the problem?
3. Is there more than one solution?

These questions are in large being answered by the theorem of existence and uniqueness of the initial value problem for first order ODE. Before we formulate this theorem we shall introduce a condition called the Lipschitz-condition.

A.2 Lipschitz-condition

The Lipschitz-condition for a function $\phi(y)$, $y \in D$ is defined as:

$$
|\phi(y_1) - \phi(y_2)| \le L|y_1 - y_2| \quad \text{for} \quad (y_1,\, y_2) \in D\,. \tag{248}
$$

where L is a constant (the Lipchitz-constant). It is easy to see that if ϕ has a derivative being limited in the region of interest, then ϕ satisfies the Lipschitz-condition.

PROOF:

From the assumption we have

$$|\phi'(y)| \leq A \qquad y \in D\,, \tag{249}$$

where A is a constant.

$$\int_{y_1}^{y_2} \phi'(y)\,dy = \phi(y_2) - \phi(y_1)\,, \tag{250}$$

$$\left|\int_{y_1}^{y_2} \phi'(y)\,dy\right| \leq \left|\int_{y_1}^{y_2} |\phi'(y)|\,dy\right| \leq \left|\int_{y_1}^{y_2} A\,dy\right| = A|y_2 - y_1|\,. \tag{251}$$

From this we obtain

$$|\phi(y_2) - \phi(y_1)| \leq A|y_2 - y_1|\,, \tag{252}$$

which is the Lipschitz-condition. It is worth noticing that the Lipschitz-condition does not require the existence of the derivative of ϕ, thus the Lipscitz-condition is a weaker condition than the existence of the derivative.

A.3 Existence and uniqueness theorem

THEOREM 1

Given the initial value problem

$$y'(x) = f(x, y(x))\,, \quad y(a) = b\,, \tag{253}$$

where a and b have given values and the function $f(x, y)$ is continuous on a region D in the (x, y) plane. Then let M be a constant such that

$$|f(x, y)| < M \quad \text{for} \quad (x, y) \in D\,. \tag{254}$$

Furthermore let $f(x, y)$ satisfy the Lipschitz-condition in y, or:

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|\,, \tag{255}$$

where A is a constant independent of x, y_1 and y_2. Let the rectangle R be defined as

$$|x - a| \leq h\,, \quad |y - b| \leq k \quad \text{where} \quad R \subset D \tag{256}$$

and $Mh < k$. Then the initial value problem, (253), has a unique solution in R that contain the point (a, b).

PROOF:

a) **Existence of solution**

First we show that a solution exists for the given conditions. Successive approximations give:

$$\begin{aligned} y_1(x) &= b + \int_a^x f(t,b)dt\,, \\ y_2(x) &= b + \int_a^x f(t,y_1(t))dt\,, \\ &\vdots \\ y_n(x) &= b + \int_a^x f(t,y_{n-1}(t))dt\,. \end{aligned}$$

We then show that $\lim_{n\to\infty}$ gives the solution we look for. The proof contain several steps.

1. *First we will show that for $a-h \le x \le a+h$, we have that $y = y_n(x)$ is located in the rectangle R, or $b-k < y < b+k$.*

   In order to prove this we use induction. If $y = y_{n-1}(x)$ is in R, then it follows that

   $$|y_n(x) - b| = \left| \int_a^x f(t,y_{n-1}(t))dt \right| \le M|x-a| \le Mh < k\,, \qquad (257)$$

   it is easy to see that this statement is correct for $n = 1$, therefore it holds for all n.

2. *Again we use induction to prove that*

   $$|y_n(x) - y_{n-1}(x)| \le \frac{MA^{n-1}}{n!}|x-a|^n. \qquad (258)$$

   Suppose this inequality is valid for $n-1$ in stead of n. Then we have

   $$y_n(x) - y_{n-1}(x) = \int_a^x \{f(t,y_{n-1}(t)) - f(t,y_{n-2}(t))\}dt\,, \qquad (259)$$

   and by using the Lipschitz-condition we find

   $$|f(t,y_{n-1}(t)) - f(t,y_{n-2}(t))| \le A|y_{n-1}(t) - y_{n-2}(t)| \qquad (260)$$

   such that from the induction hypotheses we have

   $$A|y_{n-1}(t) - y_{n-2}(t)| \le \frac{MA^{n-2}}{(n-1)!}|x-a|^{n-1}\,. \qquad (261)$$

   It then follows that

   $$|y_n(x) - y_{n-1}(x)| \le \frac{MA^{n-1}}{(n-1)!}\left| \int_a^x |t-a|^{n-1}dt \right| = \frac{MA^{n-1}}{n!}|x-a|^n. \qquad (262)$$

   For $n = 1$ we easily find

   $$|y_1(x) - b| \le \left| \int_a^x f(t,b)dt \right| \le M|x-a| \qquad (263)$$

   which is in correspondence with (258), thus the proof is completed.

3. *The sequence $y_n(x)$ converges uniformly to a limit for $a-h \le x \le a+h$.*

We consider the series

$$y_n(x) = b + \sum_{k=1}^{k=n} [y_k(x) - y_{k-1}(x)] \,, \tag{264}$$

which is an identity for finite n. The result in Eq. (262) shows that every term in the series above is less or equal to the terms in the convergent series

$$b + Mh + \frac{MA}{2!}h^2 + \cdots + \frac{MA^{n-1}}{n!}h^n + \cdots \tag{265}$$

Weierstrass M-test[12] thus shows that this series Eq. (264) converges absolute and uniformly for $|x-a| \leq h$ and $y(x) \stackrel{def}{=} \lim_{n\to\infty} y_n(x)$ exists.

4. *We have that* $y(x) \stackrel{def}{=} \lim_{n\to\infty} y_n(x)$ *satisfies the differential equation Eq. (253) with initial condition.*

   First we point out that the series for $y(x)$ is built from uniformly continuous functions such that $\lim_{n\to\infty} f(x, y_n(x)) = f(x, y(x))$. This follows from the Lipschitz-condition because

$$|\lim_{n\to\infty} f(x, y_n(x)) - f(x, y(x))| \leq A|\lim_{n\to\infty} y_n(x) - y(x)| = 0 \tag{266}$$

   and furthermore we have

$$y_n(x) = b + \int_a^x f(t, y_{n-1}(t))dt \tag{267}$$

   which in the limit $n \to \infty$ gives

$$y(x) = b + \int_a^x f(t, y(t))dt \,. \tag{268}$$

   The integrand in Eq. (268) is continuous which implies that we can take the derivative of it resulting in Eq. (253), in addition we see that $y(a) = b$, thus $y(x)$ is a solution of the initial value problem Eq. (253). It is worth noticing that this proof is constructive, which means that the technique used in the proof is a method, successive approximations or Picard iterations, that can also be used to find a solution in practice.

b) **The solution is unique**

That the solution is unique can be seen in the following way:

Let $y = y_1(x)$ and $y_2 = y_2(x)$ be two solutions of the initial value problem Eq. (253), then it follows that

[12] This test is basically the comparison test.

$$y_1(x) = b + \int_a^x f(t, y_1(t))dt \tag{269}$$

and

$$y_2(x) = b + \int_a^x f(t, y_2(t))dt \tag{270}$$

or

$$\begin{aligned} |y_1(x) - y_2(x)| &\leq \left| \int_a^x |f(t, y_1(t)) - f(t, y_2(t))|dt \right| && (271) \\ &\leq \left| \int_a^x A|y_1(t) - y_2(t)|dt \right| && (272) \\ &\leq AB|x - a|\,, && (273) \end{aligned}$$

where the constant B can be chosen as for example $2k$. We can repeat this procedure and step by step find upper bounds for

$$|y_1(x) - y_2(x)| \qquad x \in (a - h, a + h) \tag{274}$$

given as

$$\frac{A^2 B}{2!}|x - a|^2, \cdots, \frac{A^n B}{n!}|x - a|^n\,, \cdots\,. \tag{275}$$

This sequence apparently has the limit zero.[13]

Q. E. D.

This proof can easily be generalized to a system of differential equations. The essential new element is to introduce a distance (metric space) or norm which replaces the absolute value sign. This falls, however, outside the scope of these lecture notes.

B Superstable trajectories

Here follows a MAPLE-program for finding the first superstable trajectories:

```
> F1:=proc(x,lambda) 4*lambda*x*(1-x) end;
```

$$F1 := \mathbf{proc}(x,\ \lambda)\, 4 \times \lambda \times x \times (1 - x)\, \mathbf{end}$$

```
> DF1:=proc(x,lambda) diff(F1(x,lambda),x)
> end;
```

$$DF1 := \mathbf{proc}(x,\ \lambda)\, \mathrm{diff}(\mathrm{F1}(x,\ \lambda),\ x)\, \mathbf{end}$$

```
> eqa1:=F1(x,lambda)-x=0;
```

$$eqa1 := 4\,\lambda\,x\,(1 - x) - x = 0$$

[13] The account given here follows closely the book: *The theory of ordinary differential equations* by J. C. Burkill (Longman). It is not available in bookstores any more, but if you should find a used copy for sale, buy it!

```
> eqb1:=DF1(x,lambda)=0;
```

$$eqb1 := 4\,\lambda\,(1-x) - 4\,\lambda\,x = 0$$

```
> fsolve({eqa1,eqb1},{x,lambda},{x=.1..1,
> lambda=.1 ..1});
```

$$\{\lambda = .5000000000, x = .5000000000\}$$

```
> F2:=proc(x,lambda) F1(F1(x,lambda),lambda)
> end;
```

$$F2 := \mathbf{proc}(x,\ \lambda)\,\mathrm{F1}(\mathrm{F1}(x,\ \lambda),\ \lambda)\,\mathbf{end}$$

```
> eqa2:=F2(x,lambda)-x=0;
```

$$eqa2 := 16\,\lambda^2\,x\,(1-x)\,(1-4\,\lambda\,x\,(1-x)) - x = 0$$

```
> eqb2:=diff(F2(x,lambda),x)=0;
```

$$\begin{aligned} eqb2 :=\ & 16\,\lambda^2\,(1-x)\,(1-4\,\lambda\,x\,(1-x)) - 16\,\lambda^2\,x\,(1-4\,\lambda\,x\,(1-x)) \\ & + 16\,\lambda^2\,x\,(1-x)\,(-4\,\lambda\,(1-x) + 4\,\lambda\,x) = 0 \end{aligned}$$

```
> eqb2(x,lambda);
```

$$\begin{aligned} & 16\,\lambda(x,\ \lambda)^2\,(1-\mathrm{x}(x,\ \lambda))\,(1-4\,\lambda(x,\ \lambda)\,\mathrm{x}(x,\ \lambda)\,(1-\mathrm{x}(x,\ \lambda))) \\ & \quad - 16\,\lambda(x,\ \lambda)^2\,\mathrm{x}(x,\ \lambda)\,(1-4\,\lambda(x,\ \lambda)\,\mathrm{x}(x,\ \lambda)\,(1-\mathrm{x}(x,\ \lambda))) \\ & \quad + 16\,\lambda(x,\ \lambda)^2\,\mathrm{x}(x,\ \lambda)\,(1-\mathrm{x}(x,\ \lambda))\,(-4\,\lambda(x,\ \lambda)\,(1-\mathrm{x}(x,\ \lambda)) + 4\,\lambda(x,\ \lambda)\,\mathrm{x}(x,\ \lambda)) = 0 \end{aligned}$$

```
> simplify(");
```

$$\begin{aligned} & 16\,\lambda(x,\ \lambda)^2 - 128\,\lambda(x,\ \lambda)^3\,\mathrm{x}(x,\ \lambda) + 384\,\lambda(x,\ \lambda)^3\,\mathrm{x}(x,\ \lambda)^2 - 32\,\lambda(x,\ \lambda)^2\,\mathrm{x}(x,\ \lambda) \\ & \quad - 256\,\lambda(x,\ \lambda)^3\,\mathrm{x}(x,\ \lambda)^3 = 0 \end{aligned}$$

```
> subs(x=.500000, eqb2(x,lambda));
```

$$\begin{aligned} & 16\,\%2^2\,(1-\%1)\,(1-4\,\%2\,\%1\,(1-\%1)) - 16\,\%2^2\,\%1\,(1-4\,\%2\,\%1\,(1-\%1)) \\ & \quad + 16\,\%2^2\,\%1\,(1-\%1)\,(-4\,\%2\,(1-\%1) + 4\,\%2\,\%1) = 0 \\ & \%1 := .500000(.500000,\ \lambda) \\ & \%2 := \lambda(.500000,\ \lambda) \end{aligned}$$

```
> simplify(");
```

$$0 = 0$$

```
> subs(x=.500000, eqa2(x,lambda));
```

$$16\,\lambda(.500000,\,\lambda)^2\,\%1\,(1-\%1)\,(1-4\,\lambda(.500000,\,\lambda)\,\%1\,(1-\%1))-\%1=0$$
$$\%1 := .500000(.500000,\,\lambda)$$

```
> simplify(");
```

$$4.\,\lambda(\frac{1}{2},\,\lambda)^2-4.\,\lambda(\frac{1}{2},\,\lambda)^3-.5000000000=0$$

```
> fsolve({"},{lambda},{lambda=0.6..
> 0.9});
```

$$\{\lambda = .8090169944\}$$

```
> F4:=proc(x) F2(F2(x,lambda),lambda) end;
```

$$F4 := \mathbf{proc}(x)\,\mathrm{F2}(\mathrm{F2}(x,\,\lambda),\,\lambda)\,\mathbf{end}$$

```
> eqa4:=F4(x,lambda)-x=0;
```

$$eqa4 := 256\lambda^4\,x\,(1-x)\,(1-4\,\lambda\,x\,(1-x))\,(1-16\,\lambda^2\,x\,(1-x)\,(1-4\,\lambda\,x\,(1-x)))$$
$$(1-64\,\lambda^3\,x\,(1-x)\,(1-4\,\lambda\,x\,(1-x))\,(1-16\,\lambda^2\,x\,(1-x)\,(1-4\,\lambda\,x\,(1-x))))-x=0$$

```
> subs(x=.500000, eqa4(x,lambda));
```

$$256\%2^4\,\%1\,(1-\%1)\,(1-4\,\%2\,\%1\,(1-\%1))$$
$$(1-16\,\%2^2\,\%1\,(1-\%1)\,(1-4\,\%2\,\%1\,(1-\%1)))(1-64\%2^3\,\%1\,(1-\%1)$$
$$(1-4\,\%2\,\%1\,(1-\%1))\,(1-16\,\%2^2\,\%1\,(1-\%1)\,(1-4\,\%2\,\%1\,(1-\%1))))-\%1=0$$
$$\%1 := .500000(.500000,\,\lambda)$$
$$\%2 := \lambda(.500000,\,\lambda)$$

```
> simplify(");
```

$$64.\,\lambda(\frac{1}{2},\,\lambda)^4 + 57344.\,\lambda(\frac{1}{2},\,\lambda)^{12} - 98304.\,\lambda(\frac{1}{2},\,\lambda)^{13} + 8192.\,\lambda(\frac{1}{2},\,\lambda)^{11} + 7168.\,\lambda(\frac{1}{2},\,\lambda)^9$$
$$- 24576.\,\lambda(\frac{1}{2},\,\lambda)^{10} - 512.\,\lambda(\frac{1}{2},\,\lambda)^7 - 256.\,\lambda(\frac{1}{2},\,\lambda)^6 + 1792.\,\lambda(\frac{1}{2},\,\lambda)^8 - 16384.\,\lambda(\frac{1}{2},\,\lambda)^{15}$$
$$+ 65536.\,\lambda(\frac{1}{2},\,\lambda)^{14} - 64.\,\lambda(\frac{1}{2},\,\lambda)^5 - .5000000000 = 0$$

```
> fsolve({"},{lambda},{lambda=0.8..
> 0.9});
```

$$\{\lambda = .8746404248\}$$

C The universal constant α

Here is a MAPLE-program for finding an approximate value for the universal constant α

```
> g:= proc(y) 1 -a*y^2+b*y^4  end;
```

$$g := \mathbf{proc}(y)\, 1 - a \times y^2 + b \times y^4\, \mathbf{end}$$

```
> eq:= proc(y) -alpha*g(g(y/alpha)) - g(-y)
> end;
```

$$eq := \mathbf{proc}(y)\ -\alpha \times \mathrm{g}(\mathrm{g}(y/\alpha)) - \mathrm{g}(-y)\, \mathbf{end}$$

```
> asume(alpha>0);
```

$$\mathrm{asume}(0 < \alpha)$$

```
> taylor(eq(y),y=0,5);
```

$$(-\alpha\,(1 - a + b) - 1) + (-\alpha\,(2\,\frac{a^2}{\alpha^2} - 4\,\frac{b\,a}{\alpha^2}) + a)\,y^2 +$$
$$(-\alpha\,(-a\,(2\,\frac{b}{\alpha^4} + \frac{a^2}{\alpha^4}) + b\,(4\,\frac{b}{\alpha^4} + 6\,\frac{a^2}{\alpha^4})) - b)\,y^4 + \mathrm{O}(y^6)$$

```
> eq1:=-alpha*(1-a+b)-1=0;
```

$$eq1 := -\alpha\,(1 - a + b) - 1 = 0$$

```
> eq2:=-alpha*(2*a^2/alpha^2-4*b*a/alpha^2)+a=0
> ;
```

$$eq2 := -\alpha\,(2\,\frac{a^2}{\alpha^2} - 4\,\frac{b\,a}{\alpha^2}) + a = 0$$

```
> eq3:=-alpha*(-a*(2*b/alpha^4+a^2/alpha^4)+b*(
> 4*b/alpha^4+6*a^2/alpha^4))-b=0;
```

$$eq3 := -\alpha\,(-a\,(2\,\frac{b}{\alpha^4} + \frac{a^2}{\alpha^4}) + b\,(4\,\frac{b}{\alpha^4} + 6\,\frac{a^2}{\alpha^4})) - b = 0$$

```
> fsolve({eq1,eq2,eq3},{a,b,alpha},{a=0.1.
> .2.1,b=-200..200, alpha=2.  ..  3.});
```

$$\{\alpha = 2.534030095,\, b = .1276132572,\, a = 1.522241562\}$$

```
> d:=2.534030095 - 2.502907875;
```

$$d := .031122220$$

```
> pst:=(d/2.502907875)*100;
```

$$pst := 1.243442490$$