1 Boundary Layer Theory (Summary)

We summarize this by lookin at a model equation.

1.1 Outer Solution

$$\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = 2x + 1, \quad 0 < x < 1, \quad f(0) = 1,\ f(1) = 4\,. \tag{1}$$

Outer solution:

$$\frac{df^{out}}{dx} = 2x + 1 \quad \Rightarrow \quad f^{out} = x^2 + x + C\,, \quad f^{out}(1) = 4 \Rightarrow C = 2. \tag{2}$$

1.2 Inner Solution

Scaling

$$x = \delta(\epsilon)\xi, \quad \frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{1}{\delta(\epsilon)}\frac{d}{d\xi}\,, \quad \frac{d^2}{dx^2} = \frac{1}{(\delta(\epsilon))^2}\frac{d^2}{d\xi^2} \tag{3}$$

Substituting this in Eq. (1) we obtain

$$\frac{\epsilon}{(\delta(\epsilon))^2}\frac{d^2 f}{d\xi^2} + \frac{1}{\delta(\epsilon)}\frac{df}{d\xi} = 2\delta(\epsilon)\,\xi + 1\,, \tag{4}$$

If we chose a scaling so that the first term balances either of the two terms on the right hand side of Eq. (4), the resulting equation we end up with then will have the second term on the left hand side as the dominating term. Or in detail: The first choice we make is $\frac{\epsilon}{(\delta(\epsilon))^2} = O(\delta)$ or $\delta(\epsilon) = O(\epsilon^{\frac{1}{3}})$. The second choice is $\frac{\epsilon}{(\delta(\epsilon))^2} = O(1)$ or $\delta(\epsilon) = O(\sqrt{\epsilon})$. Both of these choices makes the first derivative term the dominating term, these options must therefore be rejected. The only possibility left is to balance the first and second term on the left hand side in Eq. (4), which results in $\frac{\epsilon}{(\delta(\epsilon))^2} = O(\delta(\epsilon))$ or $\delta(\epsilon) = O(\epsilon)$, so we may chose $\delta(\epsilon) = \epsilon$ as our scaling. The resulting equation is then

$$\frac{1}{\delta(\epsilon)}\left\{\frac{d^2 f}{d\xi^2} + \frac{df}{d\xi}\right\} = 2\delta(\epsilon)\,\xi + 1\,, \tag{5}$$

This scaling or *limit* is the best we can do in terms of preserving the original equation with the first term present and as much as possible of the rest of the equation. This special choice of scaling is called the *distinguished limit*[1]

Eq. (5) to leading order is

$$\left\{\frac{d^2 f^{in}}{d\xi^2} + \frac{df^{in}}{d\xi}\right\} = 0\,, \quad \Rightarrow \quad \frac{df^{in}}{d\xi} + f^{in} = C_1 \tag{6}$$

[1] In the book also called the principle of least degeneracy

And finally

$$f^{in} = C_2 e^{-\xi} + C_1\,, \quad f^{in}(0) = 1 \Rightarrow C_1 + C_2 = 1\,, \quad \text{or} \quad f^{in} = C_2 e^{-\xi} + 1 - C_2\,. \tag{7}$$

1.3 Matching

We have now established an inner solution that satisfies the boundary condition at $x = 0$, $f^{in}(0) = 1$ and an outer solution f^{out} that satisfies the boundary condition at $x = 1$, $f^{out}(1) = 4$. But how do we connect these two solutions? Prandtl invented an ingenious way to do this, called Prandtl matching condition which is formulated as

$$lim_{\xi\to\infty} f^{in}(\xi) = lim_{x\to 0} f^{out}(x)\,. \tag{8}$$

These limits can be made equal because we have a free constant C_2 in the inner solution $f^{in}(\xi)$ at our disposal. By Prandtl's matching condition, we obtain $1 - C_2 = 2$ or $C_2 = -1$. Matching the solution this way the two solutions will have a common part, determined by the value of the matching limit. In this particular case this common part is

$$lim_{\xi\to\infty} f^{in}(\xi) = lim_{x\to 0} f^{out}(x) = 2\,. \tag{9}$$

1.4 Composite expansion (solution)

We are now in a position to formulate a uniformly valid solution to leading order as

$$f(x) = f^{in}(\xi = \frac{x}{\delta(\epsilon)}) + f^{out}(x) - \text{common part} = 2 - e^{\frac{x}{\epsilon}} + x^2 + x + \mathcal{O}(\epsilon)\,. \tag{10}$$

This result compares well with the exact solution

$$f^{exact} = 2(1+\epsilon) - (1+2\,\epsilon)e^{-\frac{x}{\epsilon}} + x^2 + x(1-2\,\epsilon) = 2 - e^{-\frac{x}{\epsilon}} + x^2 + x + \mathcal{O}(\epsilon)\,. \tag{11}$$

1.5 Boundary layer thickness

An estimate of the boundary layer is the distance where $\xi = \mathcal{O}(1)$ or $x = \mathcal{O}(\delta(\epsilon))$, which in our case means $x = \mathcal{O}(\epsilon)$

1.6 Boundary layer location

It is not so that the boundary layer is always located at $x = 0$ or to the left side of the interval. Consider the example:

$$\epsilon\frac{d^2f}{dx^2}+(x-2)\frac{df}{dx}+f=0\,,\ 0<x<1\,,\ 0<\epsilon\ll 1\,,\ f(0)=3,\ f(1)=2\,. \quad (12)$$

Outer solution

$$(x-2)\frac{df^{out}}{dx}+f^{out}=0 \quad (13)$$

$$f^{out}=\frac{2}{2-x}\,,\quad f^{out}(1)=2\quad \text{(OK!)} \quad (14)$$

Scaling

$$x=\delta(\epsilon)\xi\,,\ \frac{d}{dx}=\frac{1}{\delta}\frac{d}{d\xi}\,,\ \frac{d^2}{dx^2}=\frac{1}{\delta^2}\frac{d^2}{d\xi^2}\,, \quad (15)$$

$$\frac{\epsilon}{\delta^2}\frac{d^2f^{in}}{d\xi^2}+(\delta\cdot\xi-2)\frac{1}{\delta}\frac{df^{in}}{d\xi}+f^{in}=0\,. \quad (16)$$

Checking different possibilities we find *the distinguished limit* to be $\delta=\epsilon$ resulting in

$$\frac{d^2f^{in}}{d\xi^2}-2\frac{df^{in}}{d\xi}=0\,,\Rightarrow\ \frac{df^{in}}{d\xi}-2f^{in}=C_1\Rightarrow\ f^{in}=C_2e^{\frac{1}{2}\xi}-\frac{1}{2}C_1\,, \quad (17)$$

$$f^{in}(0)=3\ \rightarrow\ C_2=3+\frac{1}{2}C_1, \quad (18)$$

and

$$f^{in}=(3+\frac{1}{2}C_1)e^{\frac{1}{2}\xi}-\frac{1}{2}C_1\,. \quad (19)$$

We have a problem because $\lim_{\xi\to\infty}e^{\frac{1}{2}\xi}$ does not exist! So this solution does not have the characteristics of a boundary layer solution. Therefore we try the other endpoint $x=1$ as an alternative position for the boundary layer. Again

$$f^{out}=\frac{C}{x-2}\quad\text{and}\quad f^{out}(0)=3\ \Rightarrow\ C=-6 \quad (20)$$

Scaling

$$(1-x)=\delta(\epsilon)\xi\,, \quad (21)$$

makes $\xi>0$ since $x<1$, and $\xi\to 0\quad\Rightarrow\quad x\Rightarrow 1$, thus

$$(1-x)=\delta(\epsilon)\xi\,,\ x<1\ \ \frac{d}{dx}=-\frac{1}{\delta}\frac{d}{d\xi}\,,\ \frac{d^2}{dx^2}=\frac{1}{\delta^2}\frac{d^2}{d\xi^2}\,, \quad (22)$$

and

$$\frac{\epsilon}{\delta^2}\frac{d^2 f^{in}}{d\xi^2} + (x-1-1)\frac{1}{\delta}\frac{df^{in}}{d\xi} + f^{in} = 0\,. \tag{23}$$

or

$$\frac{\epsilon}{\delta^2}\frac{d^2 f^{in}}{d\xi^2} + (1+\delta\,\xi)\frac{1}{\delta}\frac{df^{in}}{d\xi} + f^{in} = 0\,. \tag{24}$$

The distinguished limit is found to be

$$\frac{\epsilon}{\delta^2} \sim \frac{1}{\delta} \quad \Rightarrow \quad \delta = \epsilon\,. \tag{25}$$

We now obtain to leading order

$$\frac{d^2 f^{in}}{d\xi^2} + \frac{df^{in}}{d\xi} = 0\,. \tag{26}$$

Now the limit $\lim_{\xi\to\infty} f^{in}(\xi)$ exists and the solution have the characteristics of a boundary layer solution. We find

$$\frac{d^2 f^{in}}{d\xi^2} + \frac{df^{in}}{d\xi} = 0 \;\Rightarrow\; \frac{df^{in}}{d\xi} + f^{in} = C_1 \Rightarrow\; f^{in} = C_2 e^{-\xi} + C_1\,, \tag{27}$$

$$f^{in}(0) = 2 \;\rightarrow f^{in}(0) = C_2 + C_1 = 2, \tag{28}$$

and

$$f^{in} = (2 - C_1)e^{-\xi} + C_1\,. \tag{29}$$

We already have $f^{out} = 6/(2-x)$ and by Prandtl matching condition

$$lim_{x\to 1} f^{out}(x) = 6 = lim_{\xi\to\infty} f^{out}(\xi) = lim_{\xi\to\infty}[(2-C_1)e^{-\xi} + C_1] = C_1\,, \tag{30}$$

thus $C_1 = 6$. The uniformly valid composite solution is then given by

$$f^{comp}(x) = \frac{6}{2-x} - 4e^{-\xi} + 6 - 6 + \mathcal{O}(\epsilon) = \frac{6}{2-x} - 4e^{-\frac{1-x}{\epsilon}} + \mathcal{O}(\epsilon)\,. \tag{31}$$

This solution satisfies the boundary condition to the required order. Check this!

1.7 Some general remarks

Some general results can be obtained by study of the general linear equation

$$\epsilon\frac{d^2 f}{dx^2} + a(x)\frac{df}{dx} + b(x)f = 0 \;\; x \in [x_1,\, x_2] \;\; \&\; x_1 < x_2\,. \tag{32}$$

The critical quantity here is $a(x)$.

1. If $a(x)$ has a fixed sign in the interval $[x_1\, x_2]$ then

(a) If $a(x) > 0$ we have a boundary layer at $x = x_1$.

(b) If $a(x) < 0$ we have a boundary layer at $x = x_2$.

2. If $a(x)$ has a single zero at $x = x_0$ in the interval $[x_1\, x_2]$ $(x_1 < x_0 < x_2)$, then if $a(x_1) < 0$ we have an internal boundary layer at $x = x_0$. See Bush for more details!

More possibilities (omitted here) can be realized in the general case.