Parity violation from brane physics?

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Abstract

Studying a brane-like model for the elementary particles, where the particle masses depend on an extra spatial dimension and where the other dynamics is confined to the brane, we find that such a model predicts parity violating, massless fermions on the brane, with properties very similar to the experimentally observed parity violation in weak interactions. The model cannot describe massless scalar and vector bosons without simultaneously introducing some instability. Thus, the model does not provide a physical explanation of the confinement of a photon-like vector bosons to the brane.

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Chapter 1

Introduction

1.1 Fundamental particles and QFT

The Standard Model is really a collection of models for the interactions of fundamental particles. These particles come in two types which behave very differently:

Fermions obey the Pauli exclusion principle, which says that two of these particles cannot be in the exact same quantum state. All the material particles of the Standard Model are fermions.

Bosons can occupy the same quantum state, and thus in a way these particles behave like a classical field. This intuition is reflected in that all the mediating particles of the Standard Model are bosons.

The spin-statistics theorem gives an important correlation between the types of particles and their spin eigenvalues [2]. This theorem says that fermions have half-integral spins, while bosons have integral spins.

In the Standard Model the fundamental particles and their interactions are described by quantum field theory (QFT). In table 1.1 the fundamental particles participating in weak and electromagnetic interactions are listed, along with their spin and the name of the quantum field they are described by. Any textbook on particle physics or quantum field theory will contain a discussion of how these fields arise, see for example [9].

Table 1.1: Fundamental particles and quantum fields

Particle	Spin	Quantum field	Type
Electrons, quarks	$1/2$	Dirac	lepton
Photons	1	Maxwell	(massless) vector boson
$W^{\pm}$ and Z^0 bosons	1	Proca	vector boson

Some massive particles, such as the pion, a pseudoscalar meson [2], have spin 0, and can be described by the Klein-Gordon equation for

scalar particles. This is a rather crude approximation, as mesons are actually compound particles consisting of one quark and one antiquark. Pions should therefore be described as a bound state of fermions, but as our current knowledge of how to treat strong interactions is still quite limited, this has not yet been done. One therefore often reverts to the Klein-Gordon equation for describing such particles. However, the Higgs particle, predicted by the gauge theory which unifies weak and electromagnetic interactions, is a true scalar elementary particle [9], but the Higgs particle has still not been experimentally observed.

1.2 Weak interactions

As the name suggests, weak interactions are very elusive and short-ranged. Thus the interaction is often hard to detect even though all the fundamental particles, except photons and gluons, can participate in weak interactions. These interactions generally become important when they allow effects which are forbidden by the other interactions.

Weak interactions are mediated by the three vector bosons[1] , $W^{\pm}$ and Z^0. They have certain properties in common with the photon, which is also a vector boson, but in contrast to the latter, they have a mass of approximately 100 times the proton mass ($m_W \approx 81.8\,\mathrm{GeV}/c^2$ and $m_Z \approx 92,6\,\mathrm{GeV}/c^2$ [4]). The huge masses make the weak interactions energetically very unfavourable in comparison to electromagnetic interactions whenever the same process is possible through both interactions.

Illustrating the evasiveness of the weak interactions, according to [4], the possibility of a neutral Z^0-boson wasn't recognized until after it was suggested by Bludman in 1958. Three years later, Glashow published a paper [3] suggesting a unification of the weak and electromagnetic interactions by introducing symmetry breaking terms in the Lagrangian. Then, in 1967 and 1968 respectively, Weinberg [14] and Salam [12] proposed a gauge theory where symmetry under parity was spontanously broken. This gave a unified explanation of electromagnetic and weak interactions, as well as an explanation of why the intermediate vector bosons of weak interactions are massive, while the photon is massless. According to the Weinberg-Salam theory, processes in which the Z^0-bosons participate are very similar to electromagnetic processes, but since the Z^0-boson has a non-vanishing mass, the effect is heavily disguised behind the electromagnetic interaction. Thus, the existence of the Z^0 boson wasn't confirmed experimentally until 1983, at CERN. See [10] for a review of the experiments leading to this.

Weak interactions are responisble for the decay of massive particles into lighter ones, thus explanation why the protons and neu-

[1] As the notation suggests, the $W^{\pm}$-bosons have a charge of $\pm e$ respectively, where $e = 1.6022 \times 10^{-19}\mathrm{C}$ (in SI-units) is the absolute value of the electron charge. The Z^0-boson is electrically neutral.

trons, which make up the atomic nuclei we are surrounded by, are composed of the lightest quarks, the up and down quarks. In fact, since neutrons are slightly heavier than protons, they can decay by the weak interactions, creating a proton, an electron and a nearly massless anti-electron-neutrino,

$$n \to p + e^- + \bar{\nu}_e.$$

This is known as beta decay and is the most well known weak process.

1.3 Parity violation in weak interactions

The equations of classical mechanics are parity invariant [7, sec. 2.1.3], i.e. according to classical mechanics it is impossible to distinguish between a physical process and its mirror image. Also, strong and electromagnetic interactions are invariant with respect to parity. Until 1956 it was therefore assumed that weak interactions were also parity invariant.

However, in the early fifties, a controversy arose regarding the decay of a specific type of mesons (now known as K^+ mesons) [4, sec. 4.6]. What was believed to be two different mesons, the θ^+ and τ^+ mesons, had identical mass, spin and charge, but they decayed into states with different parity. Assuming that parity was conseved in weak decays, one had to conclude that they really were two different particles, but it was considered very bizarre that these two particles where so similar in every other respect.

In 1956, in an attempt at solving this puzzle, T. D. Lee and C. N. Yang presented an alternative interpretation of the experimental data [8]. Their suggestion was that parity was not a conserved quantity at all in weak interactions. To their surprise, they found when they worked with this hypothesis, that there had never been any experimental tests of whether parity was conserved in weak interactions. So they proposed a number of tests, one of which was carried out later the same year by C. S. Wu and her collaborators [17]. Wu and her team studied the β decay of ^{60}Co,

$$^{60}\text{Co} \to ^{60}\text{Ni} + e + \bar{\nu}_e.$$

The outcome of the experiment came as quite a surprise, as it directly confirmed the hypothesis of Lee and Yang. Wu found that in her experiment, the anti-neutrinos $\bar{\nu}_e$ allways came out with the same, right-handed helicity. The mirror process, where the anti-neutrinos would be left-handed, was not observed [2].

1.4 Models with extra dimensions

The Standard Model does not include gravity. As a quantum theory of matter, it is in conflict with the basic axioms of General Relativity of a smooth and differentable metric field as the carrier of gravity.

To resolve this difficulty, a model of the world where strings, not particles are the fundamental physical objects, have been proposed, and has become a popular candidate for a Grand Unified Theory (GUT) even though experimental verification of the theory has still not been achieved [18].

One important conceptual ingredience that string theory has introduced, is the idea that the physical world's four dimensions could emerge from a hypersurface embedded in a much larger (in terms of dimensions) space. Hypersurfaces such as these are called D-branes, or just branes, in string theory, as they in many ways resemble high-dimensional membranes. The D-branes may have more than 3 spatial dimensions, but the extra dimensions are compactified, i.e. curled up into "loops" so small that we haven't yet been able to observe them [13].

Other extra-dimensional models have also been proposed, where the interesting physics is confined to the $(3 + 1)$-dimensional space-time by means of a potential wall or a similiar well understood physical effect [11].

Chapter 2

Descripton of the model

2.1 Motivation and basic concepts

We examine a $d+1$-dimensional brane-world scenario, where we assume the directly observable space-time (the brane) is a d-dimensional subspace of the physical world. The mass parameters of the different particles are allowed to vary along the extra spatial dimension, which we, for simplicity, assume to be orthogonal to all the brane dimensions.

The idea behind our model is that the asymmetry introduced by the varying mass makes it possible to distinguish between "below" and "above" the brane, and the model might therefore provide an explanation of the parity violation observed in weak interactions.

In the Standard Model, all particle masses are assumed to be constants of the theory, where the numerical values have to be determined by experiment. In our model there are also numerical parameters related to the particle masses, which eventually will have to be determined by experiment. However, in our model, the particle masses by themselves are not scalar constants, but rather manifestations of excited physical states living in the extended space-time.

2.2 Some notes about notation

As we will be examining a space-time with an extra spatial dimension, we will use an overbar on the tensors indices to indicate that we are talking exclusively of the brane space-time. Coordinates along the extra spatial dimension will usually be denoted by the index or coordinate ρ. Thus, for an arbitrary space-time vector x^μ we may write

$$x^\mu = (x^{\bar{\mu}}, x^\rho)$$

when we want to emphasis the difference between the "normal" space-time directions $x^{\bar{\mu}}$ and the extra dimension x^ρ.

For a function $\Gamma(\rho)$ that only depends on one variable, we will also use the shorthand notation $\Gamma' = \Gamma'(\rho) = \partial_\rho \Gamma(\rho)$.

The Einstein summation convention, where repeated greek indices run over all space-time indices, will be used extensively. However, due to our choice of denoting the extra spatial dimension by ρ, this particular greek letter will never be used as a summation index.

Finally, we mention that natural units,

$$\hbar = c = 1,$$

are used throughout this report.

2.3 Free particles on the brane

We make the *ansatz* that the particles behave like free particles on the brane. Thus, we assume that for any state, the free field solution in terms of the brane dimensions factors out:

$$\Psi(t, \mathbf{x}, \rho) \propto e^{-\,\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{x})}, \tag{2.1}$$

leading to

$$\partial_t \Psi \to -\,\mathrm{i}\,\omega\Psi, \qquad \partial_t^2 \Psi \to -\omega^2\Psi, \tag{2.2}$$

$$\bar{\nabla}\Psi \to \mathrm{i}\,\mathbf{k}\Psi, \qquad \bar{\nabla}^2\Psi \to -|\mathbf{k}|^2\Psi \stackrel{\text{def}}{=} -k^2\Psi \tag{2.3}$$

and therefore

$$\bar{\Box}\Psi = -(\omega^2 - k^2)\Psi.$$

From special relativity we have the relation

$$\begin{aligned} \omega^2 = E^2 &= p^2 + m^2 \\ &= k^2 + m^2, \end{aligned} \tag{2.4}$$

recalling that we're working in natural units. In analogy with this equation, we will interpret the quantity $(\omega^2 - k^2)$ as the the mass squared of the particle, as observed from the brane.

As is common, the states in eq. (2.1) will be interpreted as representing particles or antiparticles according to the sign of ω, i.e. as particles when ω is positive and antiparticles when ω is negative.

We also make the requirement that our model must not break Lorentz invariance on the brane, so for example any effects that mark certain brane directions as special must be ruled out. This requirement will become most important when we try to describe a model for a specific magnetic interaction.

2.4 Linear mass assumption

At various occasions throughout this report we will revert to a simple, linear model, where the relation between mass M and the extra dimension ρ is of the form

$$M(\rho) = K\rho, \tag{2.5}$$

Here K is a scalar constant which must eventually be determined from experimental data.

It can be argued that this model must be interpreted as a lowest order expansion around the brane. Thus, a physically more probable relationship would be one that does not go to infinity for $\rho \to \pm\infty$, for example $M(\rho) = K \tanh(\kappa\rho)$. We will restrict our investigations to the linear relationship defined in (2.5), as we are most interested in the consequences of this model for an observer sitting on the brane.

Chapter 3

Derivations

Lagrangian field theory lies at the heart of QFT. In this theory, the principle of stationary action transforms the Lagrangian for a given field into the Euler-Lagrange equations of motion for the field. In QFT the solutions of these equations are then quantized according to the canonical quantization relation between the field representing a degree of freedom and its canonical conjugate momentum, which is just a fancy way of saying that locally the field obeys the Heisenberg uncertainty relation.

We will use this formalism to derive the results of our model for different types of particles. However, we will not gain any deeper insight into the consequences of our model by carriying out the quantization for the modes propagating along the brane, as we have allready postulated that in this subspace the particles behave as free particles. Second quantizing the fields essentially leads to the free field expansions, which are derived in standard text books on QFT, see for example [9]. The justification of the Lagrangians for the different particles can also be found in any text book on QFT.

3.1 Scalar fields

3.1.1 The Klein-Gordon equation

For scalar particles, we start with the Lagrangian for the (real) Klein-Gordon field,

$$\mathcal{L}_{\text{KG}} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - M^2\phi^2, \tag{3.1}$$

where $M = M(\rho)$ is the mass of the scalar particles in the extended space.

This Lagrangian gives the canonical momentum Π^μ_ϕ conjugate to ϕ by

$$\Pi^\mu_\phi = \frac{\partial \mathcal{L}_{\text{KG}}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$$

Using this in the Euler-Lagrange equations we are lead to the Klein-Gordon equation of motion

$$\begin{aligned} 0 &= \partial_\mu \Pi^\mu_\phi - \frac{\partial \mathcal{L}_{\text{KG}}}{\partial \phi} \\ &= \partial_\mu \partial^\mu \phi + M^2 \phi \\ &= \left[\Box + M^2\right] \phi \\ &= \left[\bar{\Box}^2 - \partial_\rho^2 + M^2\right] \phi \end{aligned}$$

Together with the free field assumption, and with $\phi \propto f(\rho)$, this becomes

$$0 = \left[-(\omega^2 - k^2) - \partial_\rho^2 + M^2\right] f(\rho). \tag{3.2}$$

Note that throughout this derivation, the mass M is a function of ρ. For convenience and readability we will usually suppress the functional dependence of M upon ρ, except when we want to avoid confusion or for other reasons deem it necessary.

3.1.2 Search for massless scalar modes

Let us assume that the function $f = f(\rho)$ satisfies

$$f \propto e^{-Q}, \tag{3.3}$$

where we will have to demand that $e^{-Q(\rho)} \to 0$ for $\rho \to \pm\infty$ to get physically acceptable, i.e. square integrable, solutions.

Inserting the last equation into (3.2) and rearranging, we get that

$$\begin{aligned} \left[\omega^2 - k^2\right] f &= \left[-\partial_\rho^2 + M^2\right] f \\ &= -\partial_\rho \left[-Q' f\right] + M^2 f \\ &= \left[-(Q'^2 - Q'') + M^2\right] f. \end{aligned}$$

This means that we will have a massless mode of any solution of this type, provided Q fullfills the relation

$$M^2 = Q'^2 - Q''. \tag{3.4}$$

for all values of ρ. This, however, seems like a rather strange constraint, as it can lead to imaginary masses in some regions of the extended space. For massless scalar particles, both the values

$$\omega = \pm k,$$

will be possible, meaning that no definite parity is prescribed on the particles, as observed from the brane.

Turning to the linear mass relation given in (2.5), we see that this does not give us any massless modes for scalar particles. The equation we get when we insert the linear relation into (3.2), is

$$\left[-\partial_\rho^2 + K^2 \rho^2\right] f(\rho) = (\omega^2 - k^2) f(\rho), \tag{3.5}$$

which is the equation for the harmonic oscillator. This means that $f(\rho)$ can be described by the harmonic oscillator eigenfunctions $\psi_n(\rho)$, see Appendix B, with the corresponding eigenvalue relation

$$M_n^2 = \omega_n^2 - k_n^2 = (2n+1)K, \quad \text{for} \quad f(\rho) \propto \psi_n(\rho) \tag{3.6}$$

We see that we can't get a scalar particle with zero mass in our model, as long as we use the linear mass assumption. The lowest possible mass (with $n = 0$ in the above relation) is given by $M_0 = \sqrt{K}$, where K can be any positive scalar.

3.2 Fermion fields

3.2.1 The Dirac equation

In relativistic quantum theory fermions are described by the Dirac equation

$$[\mathrm{i}\,\gamma^\mu \partial_\mu - M]\,\Psi = 0. \tag{3.7}$$

Here M is the mass of the fermions to be described, i is the imaginary unit, and the algebraic entities γ^μ, which are usually represented by matrices, satisfy the Dirac algebra:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}, \tag{3.8}$$

where $\eta^{\mu\nu}$ is the (inverse of the) Minkowski space-time metric. Later we will choose a specific matrix representation for the γ^μ's, and we will require the matrix corresponding to the non-brane direction ρ to be diagonal because this gives a convenient decoupling of the spinor components of Ψ, making equation (3.7) easily solvable.

To derive the Dirac equation from the principle of stationary action, we start by writing the ($d+1$-dimensional) action as

$$\mathcal{S}[\Psi, \bar{\Psi}] = \int \mathrm{d}^{d+1}x\; \mathcal{L}_{\text{Dirac}}, \tag{3.9}$$

where the Lagrangian density $\mathcal{L}_{\text{Dirac}}$ is given by

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}\left(\mathrm{i}\,\gamma^\mu \partial_\mu - M(\rho)\right)\Psi. \tag{3.10}$$

Here we have introduced a ρ-dependent mass, $M(\rho)$, which is new in comparison to the standard derivation. In the expression for $\mathcal{L}_{\text{Dirac}}$ we have also introduced the field $\bar{\Psi}$ conjugate to Ψ, defined as $\bar{\Psi} \stackrel{\text{def}}{=} \Psi^\dagger \gamma^0$.

Now, the canonical energy-momentum conjugate to $\bar{\Psi}$ is given by

$$\Pi^\mu_{\bar{\Psi}} = \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu \bar{\Psi})} = 0,$$

leading to the Euler-Lagrange equation

$$\begin{aligned} 0 &= \partial_\mu \Pi^\mu_{\bar{\Psi}} - \frac{\mathcal{L}_{\text{Dirac}}}{\partial \bar{\Psi}} \\ &= -\left[\mathrm{i}\,\gamma^\mu \partial_\mu - M\right]\Psi, \end{aligned}$$

which we recognize as the Dirac equation (3.7).

3.2.2 Iterated solution

Solutions to the Dirac equation are found by the standard method of operating from the left with the "conjugated Dirac operator", $-[\mathrm{i}\,\gamma^\nu\partial_\nu + M(\rho)]$. However, since we have now postulated that the mass M depends on ρ, we will get a new cross term in comparison with standard derivations:

$$\begin{aligned}
0 &= -\left[\mathrm{i}\,\gamma^\nu\partial_\nu + M(\rho)\right]\left[\mathrm{i}\,\gamma^\mu\partial_\mu - M(\rho)\right]\Psi \\
&= \left[\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - \mathrm{i}\,M\gamma^\mu\partial_\mu + \mathrm{i}\,\gamma^\rho M' + \mathrm{i}\,M\gamma^\nu\partial_\nu + M^2\right]\Psi \\
&= \left[\frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\nu\partial_\mu + \mathrm{i}\,\gamma^\rho M' + M^2\right]\Psi \\
&= \left[\partial_\mu\partial^\mu + \mathrm{i}\,\gamma^\rho M' + M^2\right]\Psi \\
&= \left[\Box + \mathrm{i}\,\gamma^\rho M' + M^2\right]\Psi \\
&= \left[\bar{\Box} - \partial_\rho^2 + \mathrm{i}\,\gamma^\rho M' + M^2\right]\Psi \\
&= \left[-(\omega^2 - k^2) - \partial_\rho^2 + \mathrm{i}\,\gamma^\rho M' + M^2\right]\Psi.
\end{aligned} \tag{3.11}$$

We have used the Dirac algebra (3.8), together with the fact that partial derivatives commute.

3.2.3 $2+1$ dimensions

We first want to look at the problem if the brane is one dimensional, and we proceed by choosing a specific representation for the Dirac matrices. Since our world now has only two spatial dimensions, we can use the Pauli spin matrices as a basis for the Dirac matrices:

$$\begin{aligned}
\gamma^0 = \sigma_y &= \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\
\gamma^1 = \mathrm{i}\,\sigma_x &= \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\
\gamma^\rho = \gamma^2 = -\mathrm{i}\,\sigma_z &= \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}
\end{aligned} \tag{3.12}$$

It is an easy task (see Appendix A.1) to verify that these matrices satisfy the Dirac algebra (3.8).

As stated earlier we have taken care to choose our representation such that γ^ρ has a diagonal form. This decouples the ρ-dependence from the $x^{\bar{\nu}}$-dependence in the Dirac equation, greatly simplifying the analysis.

3.2.4 Linear mass

When we insert the linear mass relation (2.5) into equation (3.11), we get that the Dirac equation becomes

$$0 = \left[-(\omega^2 - k^2) - \partial_\rho^2 + \mathrm{i}\,\gamma^\rho K + K\rho^2\right]\Psi. \tag{3.13}$$

We now assume that the ρ-dependent part of Ψ has the spinor form

$$\Psi(t,x,\rho) \propto u(\rho), \quad \text{where} \quad u(\rho) = \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix}. \tag{3.14}$$

Inserting this into equation (3.13) and using our chosen representation for γ^ρ, we get the two equations

$$\left[-(\omega^2 - k^2) - \partial_\rho^2 \pm K + K^2\rho^2\right] f_\pm(\rho) = 0$$

These can be rewritten into

$$\left[-\partial_\rho^2 + K^2\rho^2\right] f_\pm(\rho) = \left[(\omega^2 - k^2) \mp K\right] f_\pm \tag{3.15}$$

which are easily seen to be the quantum mechanical equations for two one-dimensional harmonic oscillators. This means that $f_\pm(\rho)$ can be represented by the normalized harmonic oscillator wave functions $\psi_n(\rho)$, together with corresponding eigenvalues. Comparing with the equation for the harmonic oscillator (see Appendix B), we get

$$\omega^2 - k^2 - K = (2n+1)K \qquad \text{with} \quad f_+ \propto \psi_n \tag{3.16}$$

$$\text{and} \quad \omega^2 - k^2 + K = (2n'+1)K \qquad \text{with} \quad f_- \propto \psi_{n'}, \tag{3.17}$$

recalling that $n, n' \in \{0, 1, 2, \ldots\}$. We see that consistency of the dispersion relation leads to an important relationship between the two spinor components $f_+(\rho)$ and $f_-(\rho)$, namely that

$$n' = n + 1, \tag{3.18}$$

which suggests that we can write

$$\begin{aligned} f_+(\rho) &= A_n\psi_n(\rho) \\ f_-(\rho) &= B_{n+1}\psi_{n+1}(\rho). \end{aligned} \tag{3.19}$$

Using the previously stated representation for the Dirac matrices, the Dirac equation reads

$$\begin{aligned} 0 &= \begin{pmatrix} \partial_\rho - M & \partial_t - \partial_x \\ -\partial_t - \partial_x & -(\partial_\rho + M) \end{pmatrix} \Psi \\ &\propto \begin{pmatrix} \partial_\rho - M & -\mathrm{i}(\omega + k) \\ \mathrm{i}(\omega - k) & -(\partial_\rho + M) \end{pmatrix} \begin{pmatrix} A_n\psi_n(\rho) \\ B_{n+1}\psi_{n+1}(\rho) \end{pmatrix} \\ &\propto \begin{pmatrix} \partial_\rho - K\rho & -\mathrm{i}(\omega + k) \\ \mathrm{i}(\omega - k) & -(\partial_\rho + K\rho) \end{pmatrix} \begin{pmatrix} A_n\psi_n(\rho) \\ B_{n+1}\psi_{n+1}(\rho) \end{pmatrix}, \end{aligned} \tag{3.20}$$

which gives the following equations for the relationship between the two spinor components

$$0 = A_n\psi_n'(\rho) - A_nK\rho\psi_n(\rho) - \mathrm{i}(\omega + k)B_{n+1}\psi_{n+1}(\rho) \tag{3.21}$$

$$0 = \mathrm{i}(\omega - k)A_n\psi_n(\rho) - B_{n+1}\psi_{n+1}'(\rho) - K\rho B_{n+1}\psi_{n+1}(\rho) \tag{3.22}$$

By using relations between ψ_n, ψ_n' and ψ_{n+1}, it is shown in Appendix C that this is a consistent set of equations. It is also possible to calculate the scalars A_n and B_{n+1} explicitly, but this is not particularly illuminating. The important result is that the consistency of the equations (3.21) and (3.22) means that the solution given in (3.19) is possible.

3.2.5 $4+1$ dimensions

Generalizations of our results so far to the more physically interesting scenario of a 3-dimensional brane is straightforward. Only a few points are worthy of mention.

Firstly, we still want to find a representation for the Dirac matrices that makes the Dirac equation diagonal in ρ. This is easily done by using a modified form of the Weyl representation, where we choose the gamma matrix corresponding to the ρ direction to be $\gamma^\rho = \mathrm{i}\,\gamma^5$,

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \\ \vec{\gamma} &= \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \\ \gamma^\rho = \mathrm{i}\,\gamma^5 &= \begin{pmatrix} -\mathrm{i}\,I & 0 \\ 0 & \mathrm{i}\,I \end{pmatrix}, \end{aligned} \tag{3.23}$$

with I denoting the two dimensional unit matrix. Checking that this representation obeys the Dirac algebra is only a matter of testing the behaviour of γ^ρ, since (3.23) is essentially the well-known Weyl representation. This is done explicitly in Appendix A.2.

For the $4+1$ dimensional case we assume that Ψ has the form

$$\Psi = \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix} \chi_r(\hat{\mathbf{k}}), \tag{3.24}$$

where now $f_\pm(\rho)$ are both two-component spinors. We also assume the $\chi_r(\hat{\mathbf{k}})$ to be an eigenstate for the helicity operator $(\vec{\sigma}\cdot\hat{\mathbf{k}})$, so that

$$\left(\vec{\sigma}\cdot\hat{\mathbf{k}}\right)\chi_r(\hat{\mathbf{k}}) = r\chi_r(\hat{\mathbf{k}}), \tag{3.25}$$

with $|r| = 1$.

Writing the Dirac equation out explicitly, using (3.23) and (2.1), we get

$$\begin{aligned} 0 &= \left[\mathrm{i}\left(\gamma^{\bar{\nu}}\partial_{\bar{\nu}} + \gamma^\rho\partial_\rho\right) - M\right]\Psi \\ &= \begin{pmatrix} \partial_\rho - M & \omega - \left(\vec{\sigma}\cdot\mathbf{k}\right) \\ \omega + \left(\vec{\sigma}\cdot\mathbf{k}\right) & -(\partial_\rho + M) \end{pmatrix} \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix} \chi_r(\hat{\mathbf{k}}) \\ &\propto \begin{pmatrix} \partial_\rho - M & \omega - r|\mathbf{k}| \\ \omega + r|\mathbf{k}| & -(\partial_\rho + M) \end{pmatrix} \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix}. \end{aligned} \tag{3.26}$$

We see that this gives exactly the same equations as those given by (3.20) if we make the substitutions

$$-\mathrm{i}(\omega + k) \to \omega - r|\mathbf{k}|, \tag{3.27}$$

$$\text{and} \quad \mathrm{i}(\omega - k) \to \omega + r|\mathbf{k}|. \tag{3.28}$$

All the results regarding the spinor components $f_\pm(\rho)$, can thus be taken over from the previous sections, as long as these substitutions are made throughout. Spesifically, the spinor components can be represented by solutions of the harmonic oscillator when we insert the linear mass relation.

3.2.6 Massless mode

In the spesific case where the linear mass relationship of equation (2.5) applies, an observer on the brane will see fermionic particles with mass given by

$$M_n = \sqrt{\omega^2 - k^2} = \sqrt{2K(n+1)}. \tag{3.29}$$

According to the non-negativity condition on the integers n, the ground state has a mass of $M_0 = \sqrt{2K}$. This is unfortunate, as high values of K would mean that the particle might not be observable within the range of energies available experimentally.

We therefore seek another solution of the Dirac equation, and based on the relationship (3.18), we make the *ansatz* that a spinor of the form

$$u(\rho) = \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ B_0\psi_0(\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ B_0\left(K/\pi\right)^{1/4} e^{-K\rho^2/2} \end{pmatrix} \tag{3.30}$$

is also a solution of (3.7). Inserting this into (3.20), we get the condition

$$\omega = -k, \tag{3.31}$$

rendering the mode massless. This condition also forces a definite helicity upon the massless fermions. Meanwhile, equation (3.20) also gives

$$\begin{aligned} 0 &= (\partial_\rho + K\rho)e^{-K\rho^2/2} \\ &= (-K\rho + K\rho)e^{-K\rho^2/2}, \end{aligned}$$

which ensures that (3.30) represents a possble spinor solution of the Dirac equation.

However, inserting a similarly looking spinor

$$v(\rho) = \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix} = \begin{pmatrix} A_0\psi_0(\rho) \\ 0 \end{pmatrix} = \begin{pmatrix} A_0\left(K/\pi\right)^{1/4} e^{-K\rho^2/2} \\ 0 \end{pmatrix}, \tag{3.32}$$

leads to the equation

$$\begin{aligned} 0 &= (\partial_\rho - K\rho)e^{-K\rho^2/2} \\ &= (-K\rho - K\rho)e^{-K\rho^2/2}, \end{aligned}$$

which can never be fulfilled, so such a spinor solution doesn't exist. Our model therefore effectively introduces an asymmetry on the brane for massless fermions for this particular choice of mass function.

The same programme can be carried out using the $4+1$ dimensional representation given in section 3.2.5. Inserting a spinor similar to (3.30) into (3.26), we would get a massless mode also in this model, with $r = -1$ (left-handed) for particles and $r = 1$ (right-handed) for antiparticles. This is exactly what Wu and her collaborator found for

the neutrino in 1956, see [17] and [4]. However, from experiment we know that the neutrino has a non-zero mass, so this result must be treated as a limiting case.

Using a spinor similar to (3.32) in the $4+1$ dimensional case leads to a solution which is not square integrable, just as in $2+1$ dimensions.

3.2.7 Generalizations

We can actually generalize a number of the previous results to a situation where we assume very little about the functional form of the mass function $M(\rho)$.

We starte from the iterated equation (3.11), and assume we have been able to find a representation for the Dirac matrices where

$$\gamma^\rho = \begin{pmatrix} -\mathrm{i}\, I & 0 \\ 0 & \mathrm{i}\, I \end{pmatrix},$$

and with the other Dirac matrices being off-diagonal. Here I is the identity matrix. Defining

$$\Psi \propto \begin{pmatrix} f_+(\rho) \\ f_-(\rho) \end{pmatrix},$$

we get that the iterated Dirac equation becomes

$$\left[\omega^2 - k^2\right] f_\pm(\rho) = \left[-\partial_\rho^2 \pm M' + M^2\right] f_\pm(\rho).$$

We now assume that the following spinor is a solution of the Dirac equation,

$$\Xi \propto \begin{pmatrix} 0 \\ e^{-S(\rho)} \end{pmatrix}. \tag{3.33}$$

where we must demand $e^{-S(\rho)} \to 0$ for $\rho \to \pm\infty$ if the spinor is going to be square integrable. We see from equations (3.20) and (3.26) that inserting a spinor of this type back into the original Dirac equation leads to a definite prediction of the parity for the state, i.e. through $\omega = \pm k$, where only one of the signs apply for a given state.

Inserting Ξ into the iterated Dirac equation for a massless particle, we get

$$\begin{aligned} \left[\omega^2 - k^2\right] \Xi &= \left[-\partial_\rho^2 - M' + M^2\right] \Xi \\ &= \left[S'' - S'^2 - M' + M^2\right] \Xi. \end{aligned}$$

We must therefore require

$$S'(\rho) = M(\rho),$$

to get a massless mode for such a spinor. This also ensures a square integrable solution.

Assuming symmetry between the spinor components, it is tempting to guess that another allowed spinor would be

$$\hat{\Xi} \propto \begin{pmatrix} e^{-T(\rho)} \\ 0 \end{pmatrix}, \tag{3.34}$$

From (3.20) and (3.26) we would then get that this would be represent a massless mode, and it would have a sign opposite to (3.33) in the relation between ω and k. In this sense it would be the mode conjugate to (3.33) with respect to parity.

Inserting the spinor $\hat{\Xi}$ into the iterated Dirac equation leads to the equation

$$\begin{aligned}\left[\omega^2 - k^2\right]\hat{\Xi} &= \left[-\partial_\rho + M' + M^2\right]\hat{\Xi} \\ &= \left[T'' - T'^2 + M' + M^2\right]\hat{\Xi},\end{aligned}$$

which has a massless solution only if we can identify

$$T'(\rho) = -M(\rho).$$

This, however, leads to a spinor that is not square integrable, and it can therefore not represent a physical state.

3.3 Vector fields

3.3.1 The Proca equation

The bosons mediating weak interactions, $W^\pm$ and Z^0, are vector particles. Since we seek an explanation for parity violation in such interactions it is crucial what comes out of applying our model to this type of particles.

The equations of motion for relativistic vector particles are derived from the Proca Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{Proca}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu \\ &= -\frac{1}{4}(\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu) + \frac{1}{2}M^2 A_\mu A^\mu.\end{aligned} \tag{3.35}$$

We here, as before, assume that the mass $M = M(\rho)$ is a function of the extra dimension.

From (3.35) we get that the canonical energy-momentum for A_ν is

$$\begin{aligned}\Pi^\mu_{A_\nu} &= \frac{\partial \mathcal{L}_{\text{Proca}}}{\partial(\partial_\mu A_\nu)} \\ &= -\frac{1}{2}\left[F^{\nu\mu} - F^{\mu\nu}\right] \\ &= F^{\mu\nu}.\end{aligned} \tag{3.36}$$

Thus, the Euler-Lagrange equation for A_ν is

$$\begin{aligned}0 &= \partial_\mu \Pi^\mu_{A_\nu} - \frac{\mathcal{L}_{\text{Proca}}}{\partial A_\nu} \\ &= \partial_\mu F^{\mu\nu} - M^2 A^\nu \\ &= \partial_\mu(\partial^\nu A^\mu - \partial^\mu A^\nu) - M^2 A^\nu \\ &= \partial^\nu(\partial_\mu A^\mu) - \partial_\mu \partial^\mu A^\nu - M^2 A^\nu \\ &= \partial^\nu(\partial_\mu A^\mu) - \Box A^\nu - M^2 A^\nu,\end{aligned}$$

or equivalently

$$\left[\Box + M^2\right] A^\nu = \partial^\nu(\partial_\mu A^\mu) \tag{3.37}$$

Operating from the left with ∂_ν and using that partial derivatives commute, we get

$$\begin{aligned} 0 &= \partial_\nu\partial^\nu(\partial_\mu A^\mu) - \Box(\partial_\nu A^\nu) + \partial_\nu(M^2 A^\nu) \\ &= [\Box(\partial_\mu A^\mu) - \Box(\partial_\nu A^\nu)] + \partial_\nu(M^2 A^\nu) \\ &= \partial_\nu(M^2 A^\nu) \\ &= 2MM' A^\rho + M^2\partial_\nu A^\nu, \end{aligned}$$

which we can rewrite as

$$\partial_\nu A^\nu = -2\frac{M'}{M}A^\rho, \tag{3.38}$$

provided $M \neq 0$.

We can insert this into the equations of motion, (3.37), resulting in

$$\left[\bar{\Box} - \partial_\rho^2 + M^2\right] A^\nu = -2\partial^\nu\left(\frac{M'}{M}A^\rho\right).$$

To get the equation of motion for A^ρ, we set $\nu = \rho$, and use that $\partial^\rho = -\partial_\rho$, as ρ is a spatial index. We get

$$\left[\bar{\Box} - \partial_\rho^2 + M^2 - 2\partial_\rho\frac{M'}{M}\right] A^\rho = 0. \tag{3.39}$$

Similarly, we get the equation of motion for any of the other fields,

$$\left[\bar{\Box} - \partial_\rho^2 + M^2\right] A^{\bar{\nu}} = -2\frac{M'}{M}\partial^{\bar{\nu}}A^\rho. \tag{3.40}$$

We see that the fields $A^{\bar{\nu}}$ does not in general decouple from the A^ρ component. However, we might define a gauge field χ, such that the new fields

$$\hat{A}^{\bar{\nu}} = A^{\bar{\nu}} - \partial^{\bar{\nu}}\chi \tag{3.41}$$

decouple from the A^ρ, i.e.

$$\left[\bar{\Box} - \partial_\rho^2 + M^2\right] \hat{A}^{\bar{\nu}} = 0. \tag{3.42}$$

The fields $\hat{A}^{\bar{\nu}}$ can then be represented by the free field expansions.

The necessary condition for equation (3.42) to be correct, is that χ solves the equation

$$\left[-\partial_\rho^2 + M^2\right] \chi = \partial_\rho A^\rho. \tag{3.43}$$

The reader is reminded that the Einstein summation convention does not apply on the right hand side of this equation.

Verification of the condition (3.43) can be found in Appendix D.

3.3.2 Search for massless vector modes

Having established that free field solutions exist for the vector fields, we see that we can get massless vector modes completely analogous to the scalar field by assuming

$$\hat{A}^{\bar{\nu}} \propto e^{-Q},$$

with $Q = Q(\rho)$ fulfilling

$$M^2 = Q'^2 - Q''.$$

However, the same considerations regarding imaginary masses in regions of the extended space applies here.

3.3.3 The Proca equation with linear mass

Using the linear mass relation (2.5) and assuming $\rho \neq 0$, we get

$$\partial_\rho \frac{M'}{M} f = \partial_\rho \frac{K}{K\rho} f = \partial_\rho \frac{1}{\rho} f = \left[\frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\right] f.$$

Inserting this into the equation of motion for A^ρ, eq. (3.39), we get

$$\begin{aligned} 0 &= \left[\bar{\Box} - \partial_\rho^2 + K^2\rho^2 - \partial_\rho \frac{2}{\rho}\right] A^\rho \\ &= \left[-(\omega^2 - k^2) - \partial_\rho^2 + K^2\rho^2 - \partial_\rho \frac{2}{\rho}\right] A^\rho \\ &= \left[-(\omega^2 - k^2) - \partial_\rho^2 + K^2\rho^2 - \frac{2}{\rho}\partial_\rho + \frac{2}{\rho^2}\right] A^\rho. \end{aligned}$$

which can be rewritten to

$$\left[\partial_\rho^2 + \frac{2}{\rho}\partial_\rho - \frac{2}{\rho^2} - K^2\rho^2 + \Lambda^2\right] A^\rho = 0, \tag{3.44}$$

with $\Lambda^2 \stackrel{\text{def}}{=} (\omega^2 - k^2)$.

This equation can be rewritten into the Whittaker equation, leading to solutions of the form

$$A^\rho(\rho) \propto \rho e^{-\frac{1}{2}\rho^2} \Psi(5/4 - \lambda; 5/2; \rho^2),$$

with $\lambda = \Lambda^2/4K$. Here $\Psi(a; b; x)$ is the confluent hypergeometric function [6]. See Appendix E for the transformation of (3.44) into Whittaker's equation, as well as for the derivation of the eigenvalues for Λ^2,

$$\Lambda^2 = K(4m + 5), \quad m \in \{0, 1, 2, \ldots\}.$$

We see that there is no possibility of getting a massless vector particle for arbitrary values of K using the linear mass relation.

3.4 Aside: The presence of a magnetic field

3.4.1 A model for magnetic interaction

In addition to the ρ-dependence of the particle mass, we will also do a quick investigation of how our model behaves with some sort of electromagnetic interaction. The motivation is that we want to see if an electromagnetic interaction may alter the mass of the massless solutions to get modes with small but non-zero masses.

We restrict the investigation to magnetic-like interactions, as these can alter the behaviour of the particles without being directly detectable.

To be able to use well-known expressions for electromagnetic properties, we will restrict ourselves to investigating models where the brane is either one or two dimensional. For a two dimensional brane, the magnetic field has to be oriented orthogonal to the brane if no particular direction on the brane is going to be favourized, but in such a situation particles with opposite charges will be deflected in opposite directions by the field, thus breaking Lorentz invariance on the brane. We therefore discard this model.

The other possibility is to investigate a one-dimensional brane, where the magnetic field is aligned along the brane dimension. In a world with 2 spatial dimensions, the Maxwell field tensor $F^{\mu\nu}$ takes the form

$$\begin{aligned} F^{\mu\nu} &= \partial^\nu A^\mu - \partial^\mu A^\nu \\ &= \begin{pmatrix} 0 & E^x & E^\rho \\ -E^x & 0 & B \\ -E^\rho & -B & 0 \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} F^{0x} &= E^x = \partial^x A^0 - \partial^0 A^x, \\ F^{0\rho} &= E^\rho = \partial^\rho A^0 - \partial^0 A^\rho, \\ F^{1\rho} &= B = \partial^\rho A^x - \partial^x A^\rho, \end{aligned}$$

so the magnetic field B is a (pseudo)scalar in such a system. The sign of B is defined by the handedness of the coordinate system, and is therefore a matter of convention.

Since we don't want the magnetic field to destroy Lorentz invariance on the brane, the electromagnetic field vector A^μ should not depend on the brane dimension x. We will thus take $\partial^x A^\rho = 0$. We further restrict our investigation to a system with no electric field, $E^x = E^\rho = 0$.

For simplicity we will assume that the component A^x has a linear dependence upon ρ, so we get

$$A^\mu = (A^0, A^x, A^\rho)^T = (0, a\rho, 0)^T,$$

where a is a scalar constant.

This model is interesting also because it should be possible to realize experimentally, which means that some of the results of the theory can be checked directly.

3.4.2 Scalar field with magnetic interaction

For an interaction with an electromagnetic field A^μ, the Klein-Gordon Lagrangian is modified through "minimal coupling" to become

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - M^2(\rho)\phi^*\phi.$$

Here we have introduced the covariant derivative D^μ defined by

$$D^\mu = \partial^\mu + \mathrm{i}\, A^\mu,$$

with

$$A^0 = A^\rho = 0 \quad \text{and} \quad A^x = a\rho$$

The Euler-Lagrange equations for the field ϕ^* is given by

$$\begin{aligned}
0 &= \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} - \frac{\partial\mathcal{L}}{\partial\phi^*} \\
&= \partial_\mu(D^\mu\phi) - (-\mathrm{i}\, A_\mu D^\mu\ - M^2(\rho))\phi \\
&= (\partial_\mu + \mathrm{i}\, A_\mu)D^\mu\phi + M^2(\rho)\phi \\
&= [D_\mu D^\mu + M^2(\rho)]\phi.
\end{aligned}$$

Assuming, as before, a free particle solution on the brane, and inserting the linear relationships for $A^x(\rho)$ and $M(\rho)$, we get the following equation of motion

$$\begin{aligned}
0 &= [(\partial_\mu + \mathrm{i}\, A_\mu)(\partial^\mu + \mathrm{i}\, A^\mu) + M^2(\rho)]\phi \\
&= [\partial_0\partial^0 + (\partial_x + \mathrm{i}\, A_x(\rho))(\partial^x + \mathrm{i}\, A^x(\rho)) + \partial_\rho\partial^\rho + M^2(\rho)]\phi \\
&= [\partial_t^2 - \partial_x^2 - A_x(\rho)A^x(\rho) + 2\,\mathrm{i}\, A^x(\rho)\partial_x - \partial_\rho^2 + M^2(\rho)]\phi \\
&= [\partial_t^2 - \partial_x^2 + (A^x(\rho))^2 + 2\,\mathrm{i}\, A^x(\rho)\partial_x - \partial_\rho^2 + M^2(\rho)]\phi \\
&= [-\omega^2 + k^2 + (A^x(\rho))^2 - 2A^x(\rho)k - \partial_\rho^2 + M^2(\rho)]\phi \\
&= [-\omega^2 + k^2 + a^2\rho^2 - 2a\rho k - \partial_\rho^2 + K^2\rho^2]\phi \\
&= [-\omega^2 + k^2 + (K^2 + a^2)\rho^2 - 2a\rho k - \partial_\rho^2]\phi.
\end{aligned}$$

Introducing the new variable

$$u = \sqrt{K^2 + a^2}\left(\rho + \frac{ak}{K^2 + a^2}\right)$$

gives

$$\partial_\rho^2 = (K^2 + a^2)\partial_u^2,$$

which means we can rewrite the equation of motion into

$$\begin{aligned}
0 &= [-(\omega^2 - k^2) - (K^2 + a^2)\partial_u^2 - \frac{a^2}{K^2 + a^2}k^2]\phi \\
&= (K^2 + a^2)[-\frac{\omega^2 - k^2}{K^2 + a^2} - \frac{a^2}{K^2 + a^2}k^2 - \partial_u^2 + \frac{1}{K^2 + a^2}u^2]\phi.
\end{aligned}$$

Reorganizing this a bit, we get

$$\left[-\partial_u^2 + \frac{1}{K^2+a^2}u^2\right]\phi = \frac{1}{K^2+a^2}\left[\omega^2 - (1 - \frac{a^2}{K^2+a^2})k^2\right]\phi$$

$$\Rightarrow \quad \left[-\partial_u^2 + \frac{1}{K^2+a^2}u^2\right]\phi = \frac{1}{K^2+a^2}\left[\omega^2 - \frac{(K/a)^2}{1+(K/a)^2}k^2\right]\phi$$

From our work on the harmonic oscillator, we see immediately that this is the equation for a one dimensional harmonic oscillator, and the eigenvalue equation is

$$\left[-\partial_u^2 + \frac{1}{K^2+a^2}u^2\right]\phi_n = (2n+1)\frac{1}{\sqrt{K^2+a^2}}\phi_n,$$

so, introducing the parameter $\beta = a/K$, we get the dispersion relation

$$\omega^2 = \frac{1}{1+\beta^2}k^2 + (2n+1)K\sqrt{1+\beta^2}.$$

We note that for $\beta \to 0$, we get our previous (non-interacting) result as expected, while for $|\beta| \to \infty$, the "modified speed of light",

$$c_\beta = \lim_{k\to\infty}\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{\sqrt{1+\beta^2}}$$

goes to zero. It thus seems as though the magnetic field creates an upper bound on the speed of the scalar particles which is different from the speed of light postulated in special relativity.

Chapter 4

Discussion

4.1 Behaviour of massless mode under parity

We have allready seen that our model is not parity invariant for massless fermions, according to equation (3.31). However, the free field assumption on the brane leads to a $\mathcal{CP}$ invariant theory, as we will see in this section.

We mentioned in section 2.3 that we'll interpret the fields for which $\omega > 0$ as representing particles, and those for which $\omega < 0$ as representing antiparticles. Thus the following holds for any particle field Ψ_p and antiparticle field $\Psi_{\bar{p}}$, both describing free fields,

$$\Psi_p \propto e^{-\,\mathrm{i}(|\omega|t - \mathbf{k}\cdot\mathbf{x})}, \tag{4.1}$$

$$\Psi_{\bar{p}} \propto e^{-\,\mathrm{i}(-|\omega|t - \mathbf{k}\cdot\mathbf{x})} = e^{\mathrm{i}(|\omega|t + \mathbf{k}\cdot\mathbf{x})} = e^{\mathrm{i}(|\omega|t - \mathbf{k}'\cdot\mathbf{x})}. \tag{4.2}$$

Here we have written $\mathbf{k}' \stackrel{\text{def}}{=} -\mathbf{k}$ for the antiparticle, which reflects that particles and antiparticles can be thought of as propagating in opposite directions in the temporal dimension.

Formally, the parity operator $\mathcal{P}$ corresponds to inverting the sign of the position vector $\mathbf{x}$. We can, however, also think of this transformation as a passive inversion of the frame of reference, by inverting the sign of the propagation vector $\mathbf{k}$ instead. Thus, we have

$$\mathcal{P}\Psi_p \propto e^{-\,\mathrm{i}(|\omega|t + \mathbf{k}\cdot\mathbf{x})} \tag{4.3}$$

$$\text{and} \qquad \mathcal{P}\Psi_{\bar{p}} \propto e^{\mathrm{i}(|\omega|t + \mathbf{k}'\cdot\mathbf{x})}. \tag{4.4}$$

Furthermore, we have that the charge conjugation operator $\mathcal{C}$ corresponds to Hermitean conjugation of the particle field, so we have

$$\mathcal{C}\Psi_p \propto e^{\mathrm{i}(|\omega|t - \mathbf{k}\cdot\mathbf{x})}, \tag{4.5}$$

$$\mathcal{C}\Psi_{\bar{p}} \propto e^{-\,\mathrm{i}(|\omega|t + \mathbf{k}\cdot\mathbf{x})}. \tag{4.6}$$

Combining the two operations we see that

$$\mathcal{CP}\Psi_p = \Psi_{\bar{p}} \tag{4.7}$$

$$\mathcal{CP}\Psi_{\bar{p}} = \Psi_p, \tag{4.8}$$

giving a $\mathcal{CP}$ invariant theory for fermions.

We might, however, have to relax the assumption of a free field if we assume the mass function $M(\rho)$ not to be symmetric about the brane. This could possibility lead to a theory with broken $\mathcal{CP}$ invariance.

4.2 Massless modes

The massless fermions found in section 3.2.6 inhibit a violation of parity, in the sense that the model leads to a definite relation between the signs of ω and $\mathbf{k}$ for the particles under consideration.

The scalar and vector bosons we found could be made massless only if the mass function $M(\rho)$ fulfills the relation,

$$M^2 = Q'^2 - Q''.$$

We must thus allow for the mass to become imaginary in certain regions of space. This is normally associated with an instability in the model, diverting the massless boson away from the brane. Whether these states are physically acceptable is therefore rather dubious, and it means that a stable, photon-like vector boson can probably not be deduced from our model.

However, no massless scalar particles have so far been found in Nature [1], so in this respect the result corresponds quite well with experiment.

4.3 Magnetic interaction

We introduced a magnetic interaction in our model primarily to see if this coul give the massless particles a (small) mass. We couldn't get this in an obvious way, but instead a new effect emerged, where the motion of the particle seemed to be bounded by a new maximal speed, allways less than or equal to the speed of light.

4.4 Applicability of our model to weak interactions

We would like to be able to apply our model to weak interactions, where it has been measured that the lightest fermions, the neutrinos, violate parity invariance by allways being left handed [4]. From our model we have been able to predict the correct handedness of massless fermions, but we have not been able to generalize our result to also give an explanation for fermions with a small, but non-vanishing masses.

There is good reason to believe that neutrinos have a mass, allthough

the upper bounds on these masses are very small[1]. The obvious next step would therefore be to investigate if there is a possibility of obtaining parity violating fermions with such non-zero masses. Exploiting spontaneous symmetry breaking, just as in the Weinberg-Salam theory of weak interactions, might be worth an investigation.

[1]According to [9], the upper bounds that are found from experiment are $m(\nu_e) \leq 7\,\mathrm{eV}$, $m(\nu_\mu) \leq 0.27\,\mathrm{MeV}$ and $m(\nu_\tau) \leq 35\,\mathrm{MeV}$.

Chapter 5

Conclusion

Defining a model where the masses of the elementary particles are dependent upon an extra spatial dimension, we have investigated if this model has the potential to explain the parity violation observed in weak interactions.

The results are pointing in a promising direction, as we were able to get massless fermions which violated invariance of parity, with massless fermions carrying helicity -1 (left handed) and massless anti-fermions carrying helicity 1. This is the same as is found experimentally for neutrinos in weak interactions [4].

Our model was unable to predict massless scalar or vector bosons without introducing instabilities. Thus, our model does not provide a physical explanation of why photons should be confined to the brane subspace.

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Appendix A

Verification of the Dirac algebras

A.1 $2+1$ dimensions

We here verify that the representation chosen in section 3.2.3 indeed satisfies the Dirac algebra. Recalling that the Pauli matrices satisfy the following relations [4, p. 139]

$$\sigma_i \sigma_j = \delta_{ij} + \mathrm{i}\, \epsilon_{ijk} \sigma_k, \tag{A.1}$$

where δ_{ij} and ϵ_{ijk} are, respectively, the Kronecker delta and the Levi-Cevitta symbol, we have

$$(\gamma^0)^2 = (\sigma_y)^2 = 1 \tag{A.2}$$

$$(\gamma^1)^2 = (\mathrm{i}\,\sigma_x)^2 = \mathrm{i}^2 = -1 \tag{A.3}$$

$$(\gamma^2)^2 = (-\mathrm{i}\,\sigma_z)^2 = (-1)^2\,\mathrm{i}^2 = -1 \tag{A.4}$$

$$\gamma^0\gamma^1 + \gamma^1\gamma^0 = \mathrm{i}(\sigma_y\sigma_x + \sigma_x\sigma_y) = \mathrm{i}(\mathrm{i}\,\sigma_z - \mathrm{i}\,\sigma_z) = 0 \tag{A.5}$$

$$\gamma^0\gamma^2 + \gamma^2\gamma^0 = -\,\mathrm{i}(\sigma_y\sigma_z + \sigma_z\sigma_y) = \mathrm{i}(-\,\mathrm{i}\,\sigma_x + \mathrm{i}\,\sigma_x) = 0 \tag{A.6}$$

$$\gamma^1\gamma^2 + \gamma^2\gamma^1 = -\,\mathrm{i}^2(\sigma_x\sigma_z + \sigma_z\sigma_x) = (-\,\mathrm{i}\,\sigma_y + \mathrm{i}\,\sigma_y) = 0 \tag{A.7}$$

A.2 $4+1$ dimensions

According to section 3.2.5, it is only necessary to check the anticommutation relations for γ^ρ to verify that the representation chosen in (3.23), as the rest of the representation is equal to the Weyl representation.

Performing the explicit calculations

$$\begin{aligned}\gamma^\rho\gamma^i + \gamma^i\gamma^\rho &= \mathrm{i}\left[\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\right]\\ &= \mathrm{i}\left[\begin{pmatrix}0 & -\sigma_i\\ -\sigma_i & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\right]\\ &= 0,\end{aligned} \tag{A.8}$$

$$\begin{aligned}\gamma^0\gamma^\rho + \gamma^\rho\gamma^0 &= \mathrm{i}\left[\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix} + \begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\right]\\ &= \mathrm{i}\left[\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} + \begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\right]\\ &= 0,\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}(\gamma^\rho)^2 &= \mathrm{i}^2\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\\ &= -1\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\\ &= -1,\end{aligned} \tag{A.10}$$

we see that (3.23) does indeed satisfy the Dirac algebra.

Appendix B

The quantized harmonic oscillator

Classicaly, the one-dimensional harmonic oscillator with mass m' and angular frequency ω' has a Hamiltonian H given by

$$H = \frac{p^2}{2m'} + \frac{1}{2}m'\omega'^2 q^2. \tag{B.1}$$

Quantizing this by making the canonical substitution $p \to \frac{\hbar}{\mathrm{i}}\partial_q$, and using this to solve the Schrödinger equation, $H\psi_n = E_n\psi_n$, we get the eigenvalue spectrum for the energy

$$E_n = \hbar\omega' \left(n + \frac{1}{2}\right). \tag{B.2}$$

Writing out explicitly the Schrödinger equation we get

$$\left[-\partial_q^2 + \left(\frac{m'\omega'}{\hbar}\right)^2 q^2\right] \psi_n(q) = \frac{m'\omega'}{\hbar}(2n+1)\psi_n(q), \tag{B.3}$$

where the left hand side is exactly the left hand side of equations (3.15) if we set $K = m'\omega'/\hbar$.

Appendix C

Verification of the iterated solution

From [5, p. 55] we know that the ψ_n's are given by

$$\psi_n(\rho) = \left(\frac{K}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{K}\rho) e^{-K\rho^2/2}, \tag{C.1}$$

where H_n is the n-th order Hermite polynomial. Also, we know from [5] that the Hermite polynomials satisfy the identites

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{C.2}$$

$$\text{and} \qquad H'_{n+1}(x) = 2(n+1)H_n(x) \tag{C.3}$$

Using (C.3) together with equation (C.1), we get

$$\begin{aligned}
\psi'_{n+1}(\rho) &= \frac{(K/\pi)^{1/4}}{\sqrt{2^{n+1}(n+1)!}} \left[\sqrt{K}H'_{n+1}(\sqrt{K}\rho) - K\rho H_{n+1}(\sqrt{K}\rho)\right] e^{-K\rho^2/2} \\
&= \frac{(K/\pi)^{1/4}}{\sqrt{2^{n+1}(n+1)!}} \left[2\sqrt{K}(n+1)H_n(\sqrt{K}\rho) - K\rho H_{n+1}(\sqrt{K}\rho)\right] e^{-K\rho^2/2} \\
&= \sqrt{2K(n+1)}\psi_n(\rho) - K\rho\psi_{n+1}(\rho).
\end{aligned} \tag{C.4}$$

Similarly, using (C.2) and (C.1), we get

$$\begin{aligned}
\psi_{n-1}(\rho) &= \frac{(K/\pi)^{1/4}}{\sqrt{2^{n-1}(n-1)!}} H_{n-1}(\sqrt{K}\rho) e^{-K\rho^2/2} \\
&= (K/\pi)^{1/4} \sqrt{\frac{2n}{2^n n!}} \frac{1}{2n} \left[2\sqrt{K}\rho H_n(\sqrt{K}\rho) - H_{n+1}(\sqrt{K}\rho)\right] e^{-K\rho^2/2} \\
&= \sqrt{\frac{2K}{n}}\rho\psi_n(\rho) - \sqrt{\frac{n+1}{n}}\psi_{n+1}(\rho) \qquad n \geq 1
\end{aligned} \tag{C.5}$$

which leads to

$$\begin{aligned}\psi_n'(\rho) &= \sqrt{2Kn}\psi_{n-1}(\rho) - K\rho\psi_n(\rho) \\ &= \sqrt{2Kn}\sqrt{\frac{2K}{n}}\rho\psi_n(\rho) - \sqrt{2Kn}\sqrt{\frac{n+1}{n}}\psi_{n+1}(\rho) - K\rho\psi_n(\rho) \\ &= K\rho\psi_n(\rho) - \sqrt{2K(n+1)}\psi_{n+1}(\rho) \qquad n \geq 1\end{aligned} \tag{C.6}$$

Using relation (C.4) in (3.22), we arrive at the relation

$$\begin{aligned}0 &= \mathrm{i}(\omega - k)A_n\psi_n(\rho) \\ &\quad - B_{n+1}\left(\sqrt{2(n+1)}\psi_n(\rho) - K\rho\psi_{n+1}(\rho) + K\rho\psi_{n+1}(\rho)\right) \\ &= \left[\mathrm{i}(\omega - k)A_n - \sqrt{2K(n+1)}B_{n+1}\right]\psi_n(\rho)\end{aligned} \tag{C.7}$$

Thus, a necessary condition for equation (3.19) to be valid, is that the square bracket in (C.7) is identically zero.

Turning to equation in (3.21), we use that equation (C.6) gives the following relation

$$\begin{aligned}0 &= K\rho A_n\psi_n(\rho) - \sqrt{2K(n+1)}A_n\psi_{n+1}(\rho) \\ &\quad - K\rho A_n\psi_n(\rho) - \mathrm{i}(\omega + k)B_{n+1}\psi_{n+1}(\rho) \\ &= -\left[\sqrt{2K(n+1)}A_n + \mathrm{i}(\omega + k)B_{n+1}\right]\psi_{n+1}(\rho) \qquad n \geq 1,\end{aligned} \tag{C.8}$$

which means that we must demand again that the last expression in square brackets is identically zero.

Since we have assumed $n \geq 1$ in (C.8), we must also check equation (3.21) for $n = 0$. From (C.1) we have that

$$\psi_0(\rho) = \left(\frac{K}{\pi}\right)^{1/4} e^{-K\rho^2/2} \tag{C.9}$$

$$\psi_0'(\rho) = -K\rho\left(\frac{K}{\pi}\right)^{1/4} e^{-K\rho^2/2} = -K\rho\psi_0(\rho). \tag{C.10}$$

We also have, from (C.1) and the fact that the first order Hermite polynomial is given by $H_1(x) = 2x$, that

$$\begin{aligned}2K\rho\psi_0(\rho) &= \sqrt{2K\cdot 1!}\left[\left(\frac{K}{\pi}\right)^{1/4}\frac{1}{\sqrt{2\cdot 1!}}2\sqrt{K}\rho e^{-K\rho^2/2}\right] \\ &= \sqrt{2K\cdot 1!}\left[\left(\frac{K}{\pi}\right)^{1/4}\frac{1}{\sqrt{2\cdot 1!}}2H_1(\sqrt{K}\rho)e^{-K\rho^2/2}\right] \\ &= \sqrt{2K}\psi_1(\rho)\end{aligned} \tag{C.11}$$

Inserting this into (3.21), we get

$$\begin{aligned}0 &= -2K\rho A_0\psi_0(\rho) - \mathrm{i}(\omega + k)B_1\psi_1(\rho) \\ &= -\left(\sqrt{2K}A_0 + \mathrm{i}(\omega + k)B_1\right)\psi_1(\rho),\end{aligned} \tag{C.12}$$

which is exactly the relation found in (C.8), proving that (C.8) and the argument following it is also valid for $n = 0$.

Combining (C.7) and (C.8), we get

$$\begin{aligned} 0 &= \mathrm{i}(\omega - k)A_n - \sqrt{2K(n+1)}B_{n+1} \\ 0 &= \mathrm{i}(\omega + k)B_{n+1} + \sqrt{2K(n+1)}A_n. \end{aligned} \tag{C.13}$$

Multiplying the first term by $\mathrm{i}(\omega + k)$ and the last by $\sqrt{2K(n+1)}$, we get

$$\begin{aligned} 0 &= -(\omega + k)(\omega - k)A_n - \mathrm{i}\sqrt{2K(n+1)}(\omega + k)B_{n+1} \\ 0 &= 2K(n+1)A_n + \mathrm{i}\sqrt{2K(n+1)}(\omega + k)B_{n+1}, \end{aligned} \tag{C.14}$$

which, when we add the two equations, becomes

$$\begin{aligned} 0 &= [2K(n+1) - (\omega^2 - k^2)]A_n, \\ \Rightarrow \quad & (\omega^2 - k^2) = 2K(n+1) \end{aligned} \tag{C.15}$$

As we can se, this is exactly the relation we found in (3.16).

Appendix D

The gauge condition for the Proca fields

The gauge field χ is chosen so that

$$\hat{A}^{\bar{\nu}} = A^{\bar{\nu}} - \partial^{\bar{\nu}}\chi \tag{D.1}$$

solves the equation

$$\left[\bar{\Box} - \partial_\rho^2 + M^2\right] \hat{A}^{\bar{\nu}} = 0.$$

Substituting (D.1) into the last equation we get that the following relation holds for χ:

$$\begin{aligned}\left[\bar{\Box} - \partial_\rho^2 + M^2\right] (A^{\bar{\nu}} - \partial^{\bar{\nu}}\chi) &= 0 \\ \left[\bar{\Box} - \partial_\rho^2 + M^2\right] \partial^{\bar{\nu}}\chi &= \left[\bar{\Box} - \partial_\rho^2 + M^2\right] A^{\bar{\nu}}.\end{aligned}$$

Using the equation of motion for $A^{\bar{\nu}}$, equation (3.40), we can transform this into

$$\begin{aligned}\left[\bar{\Box} - \partial_\rho^2 + M^2\right] \partial^{\bar{\nu}}\chi &= -\partial^{\bar{\nu}} 2\frac{M'}{M} A^\rho \\ \left[\bar{\Box} - \partial_\rho^2 + M^2\right] \chi &= -2\frac{M'}{M} A^\rho,\end{aligned} \tag{D.2}$$

where we have cancelled the (commuting) operator $\partial^{\bar{\nu}}$ in the last step.

We now turn to the equation of motion for A^ρ:

$$\left[\bar{\Box} - \partial_\rho^2 + M^2\right] A^\rho = 2\partial_\rho \frac{M'}{M} A^\rho.$$

Operating on this with ∂_ρ from the left, we derive

$$\begin{aligned}\partial_\rho \left[\bar{\Box} - \partial_\rho^2 + M^2\right] A^\rho &= 2\partial_\rho^2 \frac{M'}{M} A^\rho \\ \left[\bar{\Box} - \partial_\rho^2 + M^2\right] \partial_\rho A^\rho + 2MM' A^\rho &= \partial_\rho^2 \left(2\frac{M'}{M} A^\rho\right) \\ \left[\bar{\Box} - \partial_\rho^2 + M^2\right] \partial_\rho A^\rho &= \left[\partial_\rho^2 - M^2\right] 2\frac{M'}{M} A^\rho \\ \left[\bar{\Box} - \partial_\rho^2 + M^2\right] \partial_\rho A^\rho &= \left[-\partial_\rho^2 + M^2\right] \left(-2\frac{M'}{M} A^\rho\right).\end{aligned}$$

Substituting for the rightmost term with the help of equation (D.2), we arrive at

$$\begin{aligned}\left[\bar{\Box} - \partial_\rho^2 + M^2\right](\partial_\rho A^\rho) &= \left[-\partial_\rho^2 + M^2\right]\left[\bar{\Box} - \partial_\rho^2 + M^2\right]\chi \\ &= \left[\bar{\Box} - \partial_\rho^2 + M^2\right]\left[-\partial_\rho^2 + M^2\right]\chi,\end{aligned}$$

since the operator $\bar{\Box}$ commutes with $\left[-\partial_\rho^2 + M^2\right]$, and also any operator commutes with itself. Thus we get the condition (3.43) on the gauge field χ:

$$\left[-\partial_\rho^2 + M^2\right]\chi = \partial_\rho A^\rho$$

Appendix E

Whittaker's equation

Equation (3.44) can be transformed into Whittaker's equation by the following substitutions. Changing variables by setting

$$x = K\rho^2,$$

we get

$$\begin{aligned} & \partial_\rho = 2K\rho\partial_x, \\ \Rightarrow \quad & \frac{2}{\rho}\partial_\rho = 4K\partial_x = 4Kx\left(\frac{1}{x}\partial_x\right) \\ \text{and} \quad & \partial_\rho^2 = 2K\left(\partial_x + 2K\rho^2\partial_x^2\right) = 4Kx\left(\partial_x^2 + \frac{1}{2x}\partial_x\right). \end{aligned}$$

Substituting this into (3.44), leads to

$$\begin{aligned} 0 &= 4Kx\left[\left(\partial_x^2 + \frac{1}{2x}\partial_x\right) + \frac{1}{x}\partial_x - \frac{1}{2x^2} - \frac{1}{4} + \frac{\Lambda^2}{4Kx}\right]A^\rho \\ &= 4Kx\left[\partial_x^2 + \frac{3}{2x}\partial_x - \frac{1}{4} + \frac{\Lambda^2/4K}{x} - \frac{1/2}{x^2}\right]A^\rho. \end{aligned}$$

We now want to get rid of the first order partial derivative with respect to x. We do this by finding functions f, g and ψ such that $A^\rho = g\psi$, and requiring that

$$g\left[\partial_x^2 + f\right]\psi = \left[\partial_x^2 + \frac{3}{2x}\partial_x\right]g\psi,$$

where f is any function not containing differential operators. Expanding the last equation, we get

$$\begin{aligned} g\left[\partial_x^2 + f\right]\psi &= \left[\partial_x^2 + \frac{3}{2x}\partial_x\right]g\psi \\ &= \left[g'' + 2g'\partial_x + g\partial_x^2\right]\psi + \frac{3}{2x}\left[g' + g\partial_x\right]\psi \\ &= \left[\left(g'' + \frac{3}{2x}g'\right) + \left(2g' + \frac{3}{2x}g\right)\partial_x + g\partial_x^2\right]\psi, \end{aligned}$$

where $g' = \partial_x g$.

We see that we must demand

$$\begin{aligned} 0 &= 2g' + \frac{3}{2x}g \\ \Rightarrow \quad g' &= -\frac{3}{4x}g \\ \Rightarrow \quad g'' &= \frac{3}{4x^2}g - \frac{3}{4x}g' = \frac{21}{16}\frac{1}{x^2}g. \end{aligned}$$

The first equation can easily be solved, leading to $g = x^{-3/4}$. We also see that we must have

$$gf = g'' + \frac{3}{2x}g' = g\frac{3}{16}\frac{1}{x^2}.$$

Inserting this into (3.44), we get Whittaker's equation [6] for ψ,

$$\partial_x^2\psi + \left[-\frac{1}{4} + \frac{\Lambda^2/4K}{x} + \frac{1/4 - (3/4)^2}{x^2}\right]\psi = 0. \tag{E.1}$$

Writing $\lambda = \Lambda^2/4K$ and $\mu = 3/4$, this has solutions of the form

$$\begin{aligned} \psi &\propto M_{\lambda,\mu}(x) \\ &= x^{1/2+\mu}e^{-1/2x}\Phi(\mu - \lambda + 1/2; 2\mu + 1; x). \end{aligned}$$

$\Phi(a; b; z)$ is known as the confluent hypergeometric function [15], and is given by the power series

$$\Psi(a; b; z) = \sum_{n=0}^{\infty} P_n z^n = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\, n!} z^n,$$

where $(c)_n$ denotes $c(c+1)(c+2)\ldots(c+(n-1)) = \Gamma(c+n)/\Gamma(c)$, known as the Pochhammer symbol [16]. See [6] for a nice discussion of the confluent hypergeometric function.

In our case

$$\begin{aligned} a &= \mu - \lambda + \frac{1}{2} = \frac{5}{4} - \lambda = \frac{5}{4} - \frac{\Lambda^2}{4K}, \\ b &= \frac{5}{2}. \end{aligned}$$

The asymptotic behaviour of the expansion coefficienst P_n, is given by

$$\frac{P_{n+1}}{P_n} = \frac{(a+n)}{(n+1)(b+n)} = \frac{(1+a/n)}{(n+1)(1+b/n)} \overset{n\to\infty}{\to} \frac{1}{n},$$

and since this is the same behaviour as the expansion coefficients for the exponential function,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n,$$

we will have that for large n, the hypergeometric function Ψ goes like the exponential function. This means that

$$\begin{aligned} A^\rho &\propto x^{-3/4} x^{1/2} x^{3/4} e^{-1/2x} \Phi(5/4 - \lambda; 5/2; x) \\ &\overset{\text{large } n}{\to} x^{1/2} e^{-1/2x} e^x = x^{1/2} e^{1/2x}, \end{aligned}$$

which diverges for $x \to \infty$, and therefore cannot describe a physical state. To get a square integrable function, the power series of Φ must break off into a polynomial of finite order. Since b is not a negative integer, $(b)_n$ will never become zero, and therefore the denominator of P_n does not lead to any singulatities. However, the break-off condition means that $P_{m+1} = 0$ for some positive integer m, i.e. that

$$0 = a + m = \frac{5}{4} - \lambda + m = \frac{5}{4} - \frac{\Lambda^2}{4K} + m \quad \text{for} \quad m \in \{0, 1, 2, \ldots\},$$

which translates into the eigenvalue relation

$$\Lambda^2 = M_m^2 = K(4m + 5). \tag{E.2}$$

For completeness, we also state again the solution of Whittaker's equation for the field A^ρ,

$$A^\rho \propto \sqrt{K}\rho e^{-\frac{K\rho^2}{2}} \Phi\left(\frac{5K - \Lambda^2}{4}; 5/2; K\rho^2\right). \tag{E.3}$$