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Corrected operator splitting for a 2×2 system of parabolic equations

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Outline

- Introduction
- The Front Tracking Method
- The Corrected Operator Splitting Method
- Numerical experiments:
  A Triangular Three Phase Flow Model
- Summary

Introduction

Consider a (1D) Cauchy problem for a 2×2 system of parabolic equations

$$\begin{aligned} \partial_t U + \partial_x F(U) &= \varepsilon \partial_x^2 D(U), \\ U(x,0) &= U_0(x), \quad x \in \mathbb{R},\ t > 0. \end{aligned} \tag{1}$$

We will solve this system using an operator splitting method.

COS algorithm

Solve the hyperbolic equation

$$\partial_t U + \partial_x F(U) = 0, \quad U(x,0) = U_0(x).$$

Solve the diffusion equation

$$\partial_t V + \partial_x F^{\mathrm{res}}(x,V) = \varepsilon \partial_x^2 D(V), \quad V(x,0) = U(x,\Delta t)$$

where $F^{\mathrm{res}}(x,V)$ is to be constructed in the following.

Why corrected operator splitting?

The temporal *splitting error* in OS can be significant in regions containing steep gradients (fronts).

The corrected operator splitting (COS) was developed to amend this shortcoming.

The COS approach: The wave structure from the convection step is used to identify where the splitting error in OS occurs.

This (potential) splitting error is then compensated for in the diffusion step.

Solvers

Hyperbolic solver: A front tracking method.

Diffusion solver: A central finite difference scheme.

The front tracking method constitutes an important part of our solution strategy since it implicitly generates the residual flux $F^{\mathrm{res}}(x, U)$.

A Central Finite Difference Scheme

Consider the parabolic system

$$\partial_t U + \partial_x F^{\mathrm{res}}(x, U) = \partial_x^2 D(U) = 0, \quad U(x, 0) = U_0(x).$$

The central finite difference scheme is given

$$\begin{aligned} U_j^{n+1} = U_j^n &+ \frac{\varepsilon \Delta t}{(\Delta x)^2}\Big(D(U_{j+1}^n) - 2D(U_j^n) + D(U_{j-1}^n)\Big) \\ &- \frac{\Delta t}{2\Delta x}\Big(F^{\mathrm{res}}(x_{j+1}, U_{j+1}^n) - F_1^{\mathrm{res}}(x_{j-1}, U_{j-1}^n)\Big)\kappa(j-1, j+1) \end{aligned}$$

where κ is one if both x_{j-1} and x_{j+1} are in the same domain r_i and zero otherwise.

The Front Tracking Method

Consider the 2×2 system of conservation laws

$$\partial_t U + \partial_x F(U) = 0, \quad U(x,0) = U_0(x). \qquad (2)$$

Front tracking is an algorithm for computing a piecewise constant approximation to (2). A *front* is a jump discontinuity.

Scalar problems: Finite number of steps, even in infinite time.

General systems: Data reduction must be invoked to prevent an infinite buildup of fronts.

The solution consists of a sequences of fronts (shocks) U_{i-1}/U_1 with shock speed σ_i, $i = 1, \ldots, n$.

The algorithm goes as follows:

1. First, u_0 is approximated by a step function.

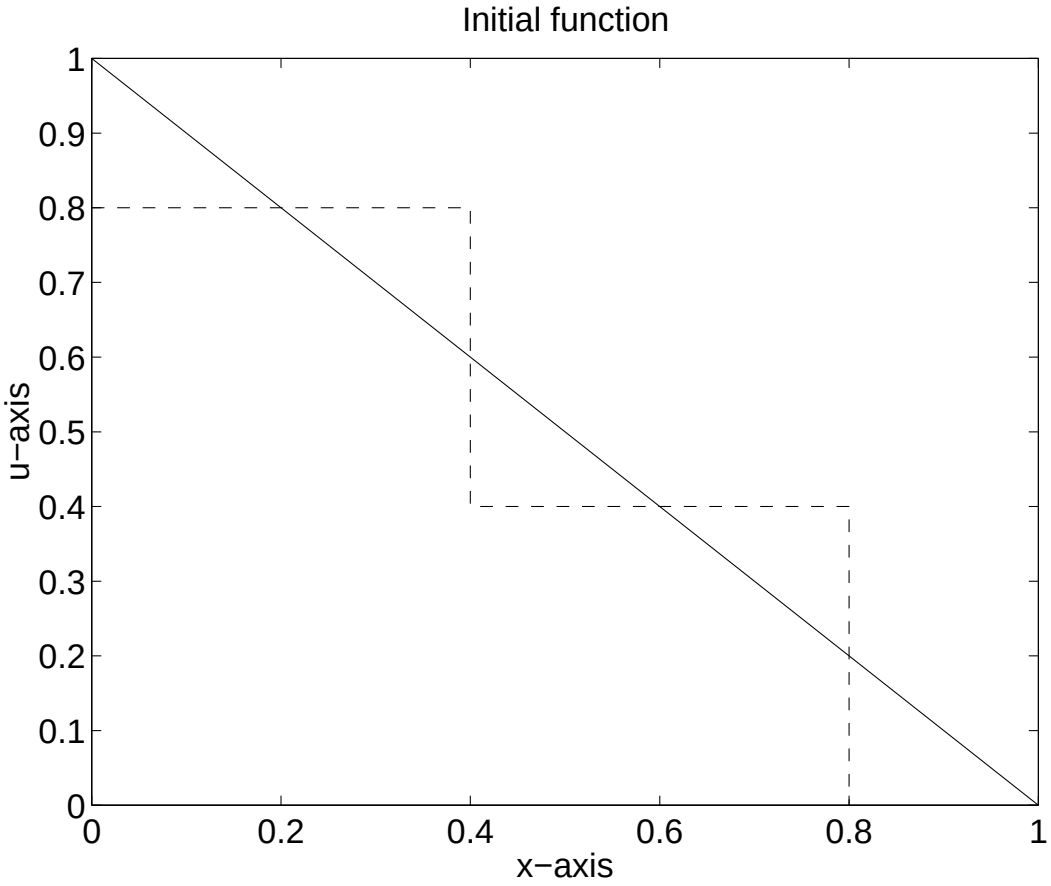
2. A Riemann problem is associated with each jump in the approximate initial data.

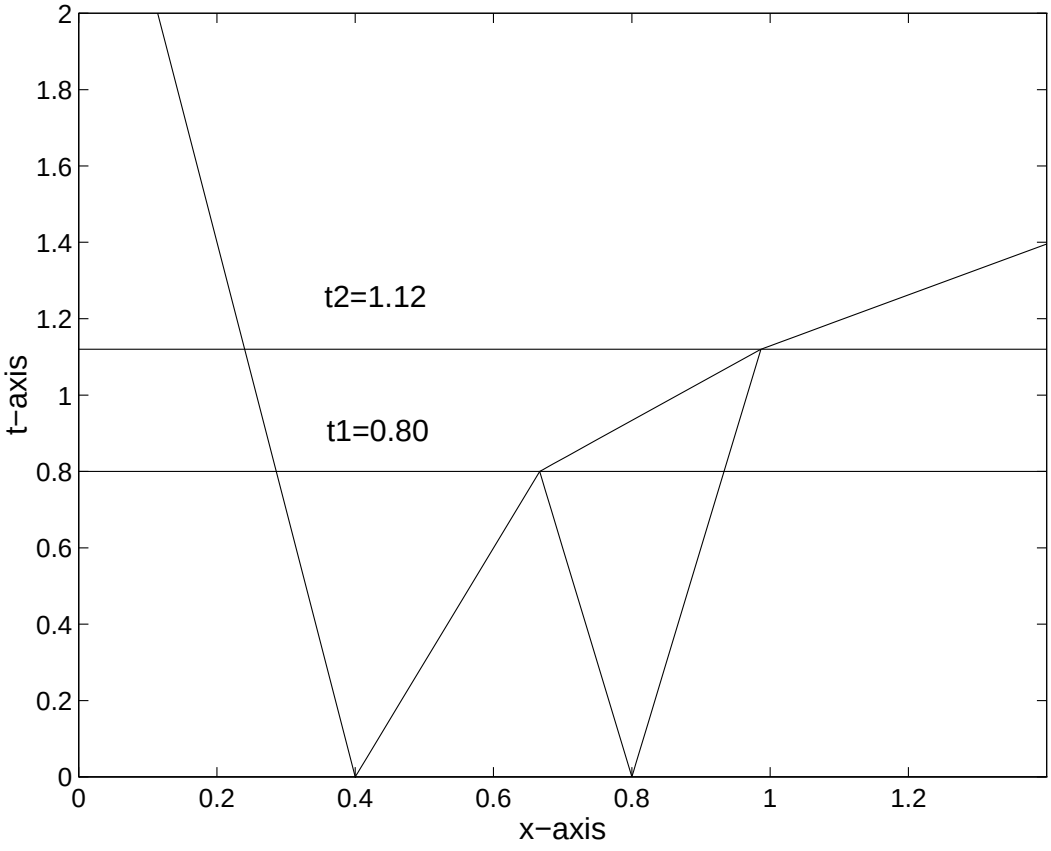
3. All Riemann problems are solved. Rarefraction waves are approximated by a series of small fronts. In this way, each Riemann problem produces a sequence of fronts that travel with a finite wave speed.

4. All fronts are *tracked*. Colliding fronts define new Riemann problems which are solved. Colliding fronts are removed and fronts from the new Riemann problems are added.

5. The tracking ends when there are now more collision before final time, or final time is reached.

Front tracking





The Corrected Operator Splitting Method

Consider Eq. (1).

1) Solve

$$\partial_t U + \partial_x F(U) = 0, \quad U(x,0) = U_0(x). \tag{3}$$

2) Find F^{res} generated by the front tracker solution of 1).

3) Solve

$$\partial_t V + \partial_x F^{\mathrm{res}}(x,V) = \varepsilon \partial_x^2 D(V), \quad V(x,0) = U(x,\Delta t) \tag{4}$$

Scalar COS

Consider the scalar parabolic initial value problem

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 d(u), \quad u(x,0) = u_0(x). \tag{5}$$

The corrected operator splitting for Eq. (5) is given by

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,0) = u_0(x), \tag{6}$$

$$\partial_t v + \partial_x f^{\mathrm{res}}(v,x) = \partial_x^2 d(v), \quad v(x,0) = u(x,\Delta t). \tag{7}$$

The concept of a residual flux was first introduced by Espedal and Ewing in the context of two phase flow. This idea has been developed further amongst others by Dahle, Karlsen and Risebro.

Residual flux for scalar equations

Convex envelope: If a shock develops in the hyperbolic step of OS, the solution of the pure hyperbolic equation is governed by the convex envelope f_c of f. The residual flux is then given by

$$f^{\mathrm{res}} = f - f_c.$$

Generalization: The local behaviour of the solution around the i'th shock is well approximated by a linearized law

$$\partial_t u + \partial_x(\sigma_i u) = 0, \quad u(x_0, t_0) = \begin{cases} u^{\mathrm{l}}, & x < x_i, \\ u^{\mathrm{r}}, & x > x_i. \end{cases}$$

The error introduced in the OS algorithm can now be understood in terms of the difference between equation (6) and the linearized equation. Then the residual flux is given by

$$\partial_x(f^{\mathrm{res},i}) = \partial_x(f(u) - \sigma_i u).$$

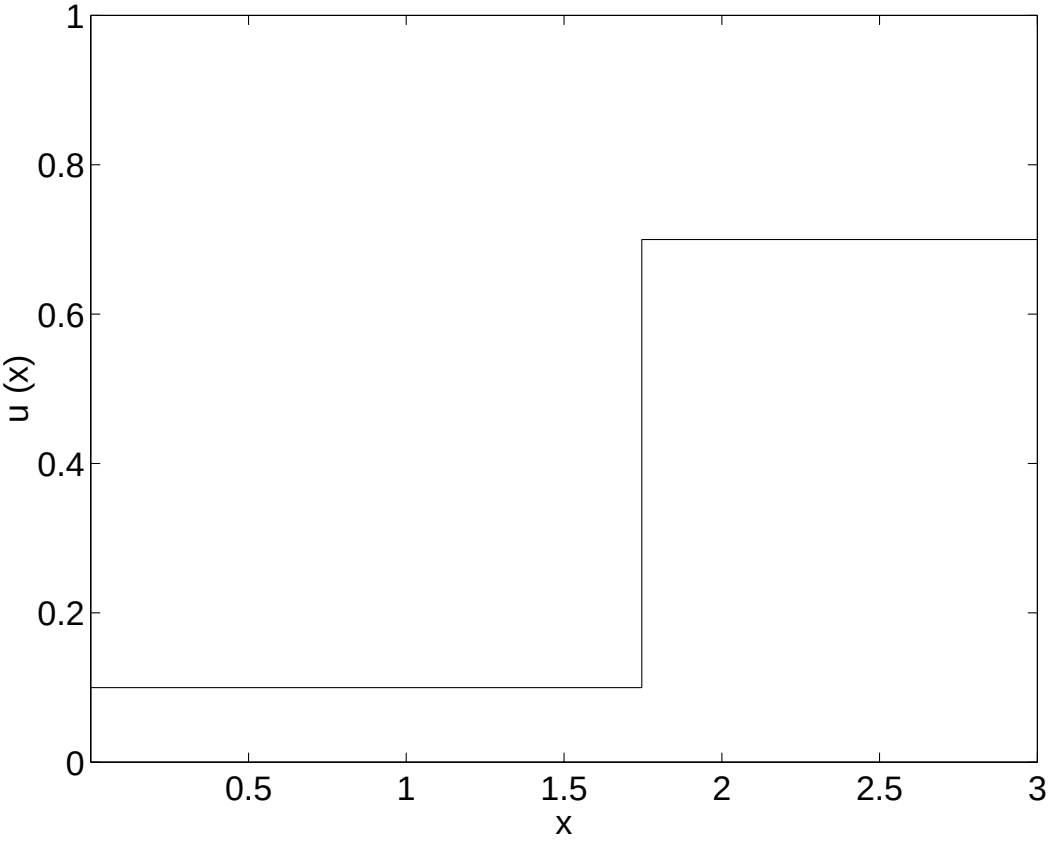
By integrating the expression for the residual flux with respect to x, we get

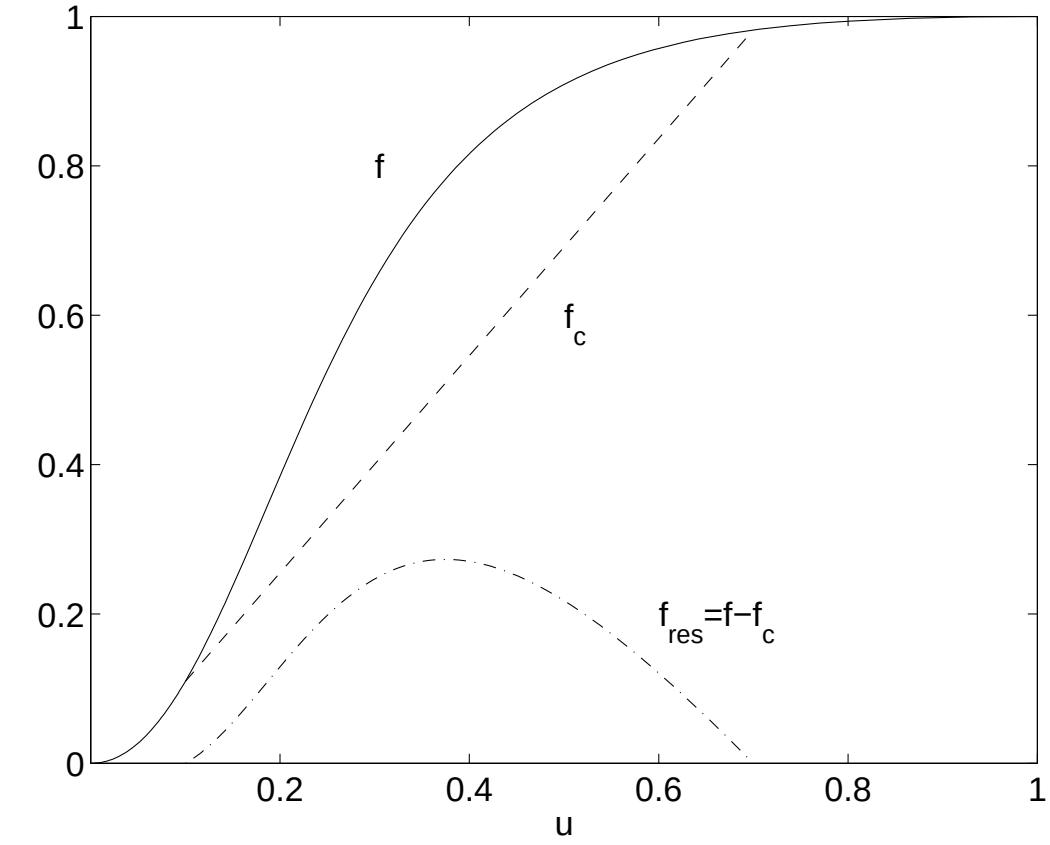
$$f^{\mathrm{res},i}(u) = (f(u) - f(u^{\mathrm{l}})) - \sigma_i(u - u^{\mathrm{l}}).$$

We define the residual flux $f^{\mathrm{res},i}$ associated with the i'th shock as

$$f^{\mathrm{res},i}(u) = \begin{cases} (f(u) - f(u^{\mathrm{l}})) - \sigma_i(u - u^{\mathrm{l}}), & u \in (u^{\mathrm{l}}, u^{\mathrm{r}}), \\ 0, & \text{otherwise.} \end{cases}$$

Residual flux





Residual flux for 2×2 systems

Approximate system (3) by the linearized system of conservation laws

$$\partial_t U + \partial_x(\sigma_i U) = 0, \quad U(x_0, t_0) = \begin{cases} U^{\mathrm{l}}, & x < x_i, \\ U^{\mathrm{r}}, & x > x_i. \end{cases} \tag{8}$$

Subtracting this equation from Eq. (3) and integrating gives the residual flux $F^{\mathrm{res},i}$:

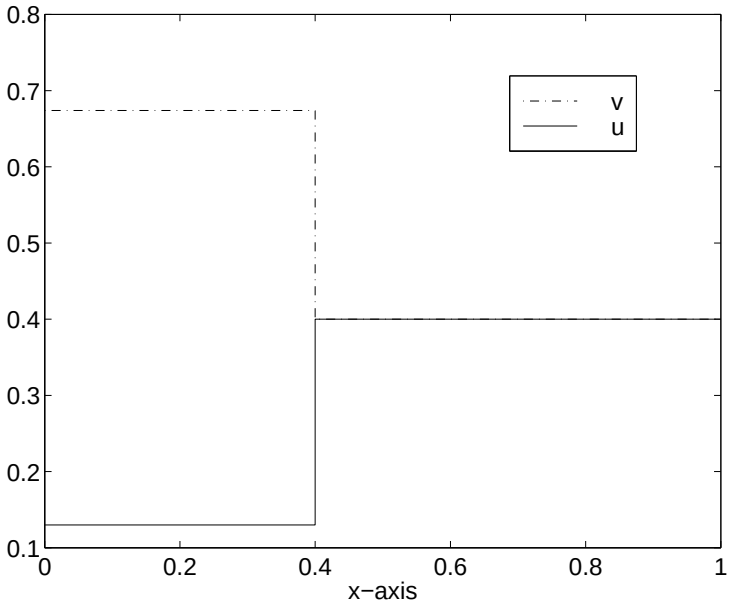
$$F^{\mathrm{res},i}(U) = \begin{cases} (F(U) - F(U^{\mathrm{l}})) - \sigma_i(U - U^{\mathrm{l}}), & U \in (u_1^{\mathrm{l}}, u_1^{\mathrm{r}}) \times (u_2^{\mathrm{l}}, u_2^{\mathrm{r}}), \\ 0, & \text{otherwise.} \end{cases}$$

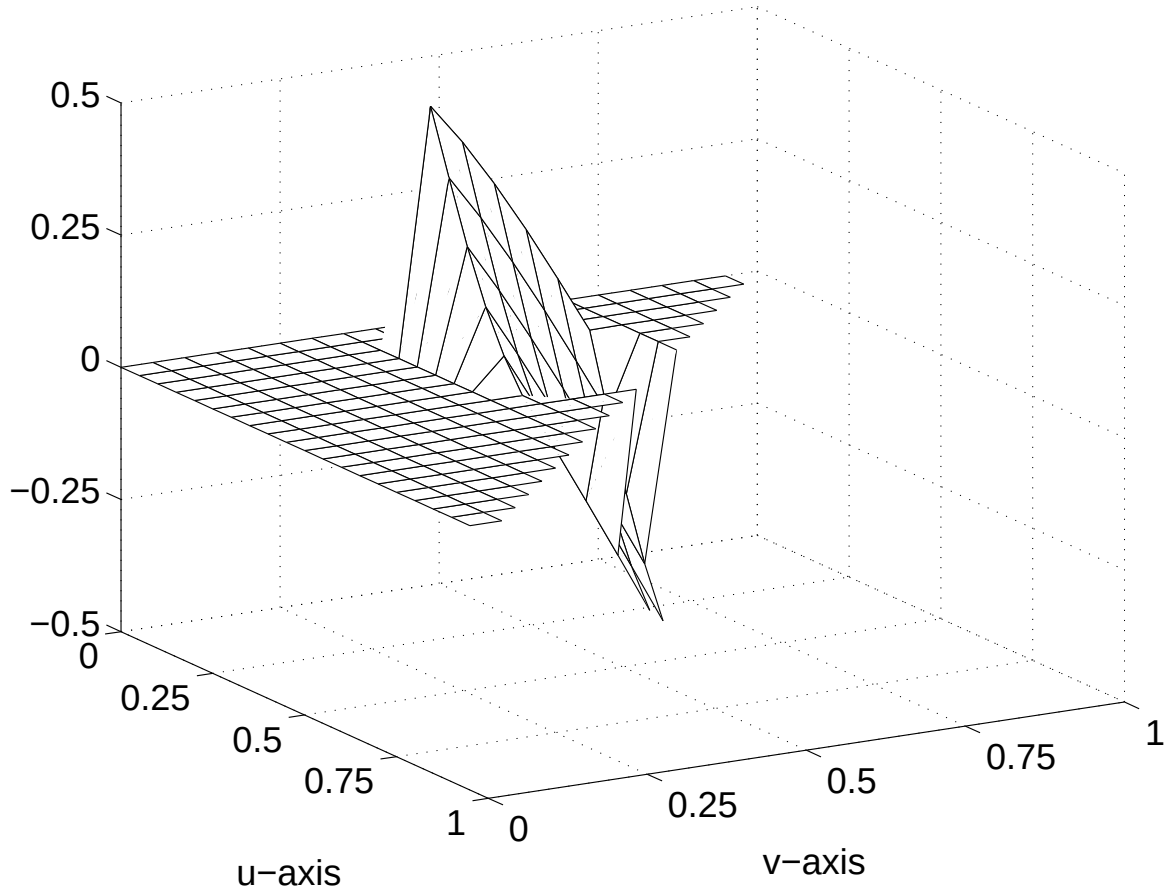
Define

$$F^{\mathrm{res}}(x,U)=\sum_{i=0}^{n}F^{\mathrm{res},i}(U)\mathcal{I}_{r_i}(x). \tag{9}$$

- r_i are domains around each shock satisfying $r_i \cap r_j = \emptyset$ whenever $i \neq j$.
- The domains r_i have to be small since the approximation (8) is only local.

Residual flux for systems





Numerical Experiments

A Triangular Three Phase Flow Model

Consider the initial value problem for the following 2×2 system of parabolic equations:

$$\begin{aligned} \partial_t u + \partial_x f(u) &= \varepsilon \partial_x^2 d(u), \\ \partial_t v + \partial_x g(u, v) &= \varepsilon \partial_x^2 d(v), \end{aligned} \tag{10}$$

u , v: Phase saturations (gas and water)

f , g: Fractional flow functions

d: Diffusion function - nonlinear.

The Jacobian matrix of the fractional flow functions is (lower) triangular, hence a triangular system.

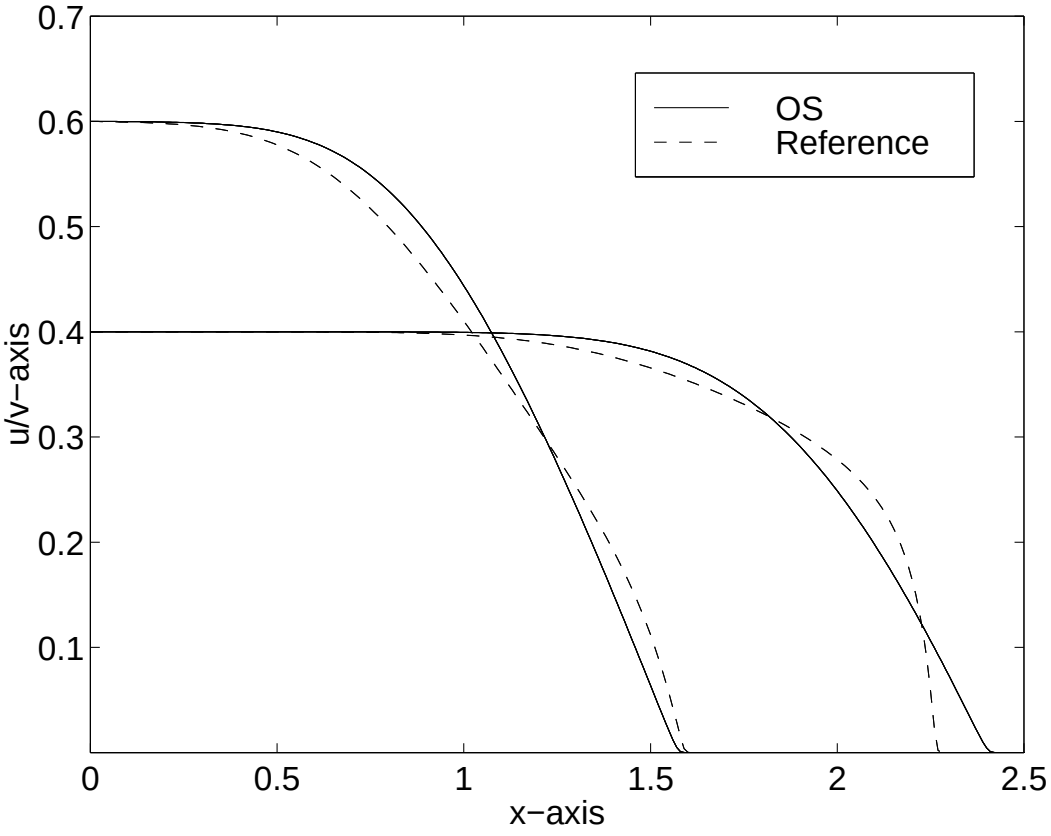
Ex. 1: Single Riemann problem

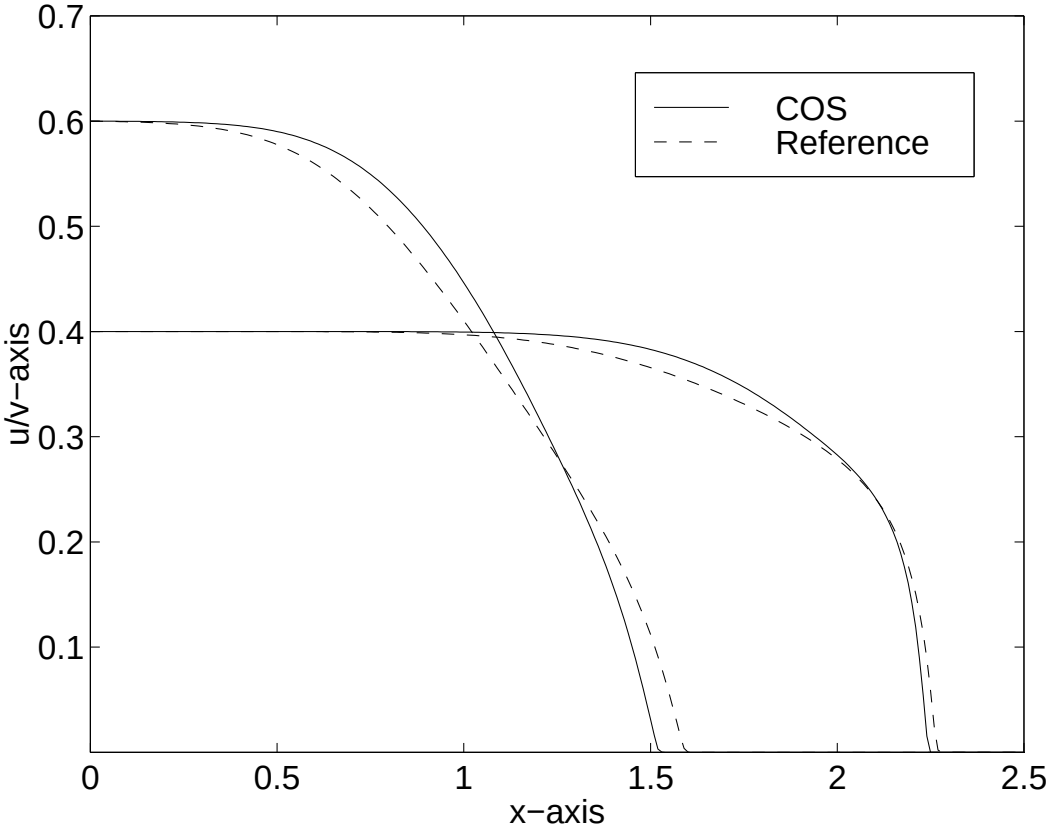
Initial data

$$u_0 = \begin{cases} 0.4 & \text{if} \quad x < 1.0, \\ 0.0 & \text{if} \quad x > 1.0, \end{cases} \quad \text{and}$$

$$v_0 = \begin{cases} 0.6 & \text{if} \quad x < 1.0, \\ 0.0 & \text{if} \quad x > 1.0. \end{cases}$$

$\epsilon = 0.1$, $t = 0.5$, $\Delta t = 0.5$ - one time step.


OS
Reference
u/v−axis
x−axis
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
2.5


COS
Reference
u/v−axis
x−axis
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
2.5

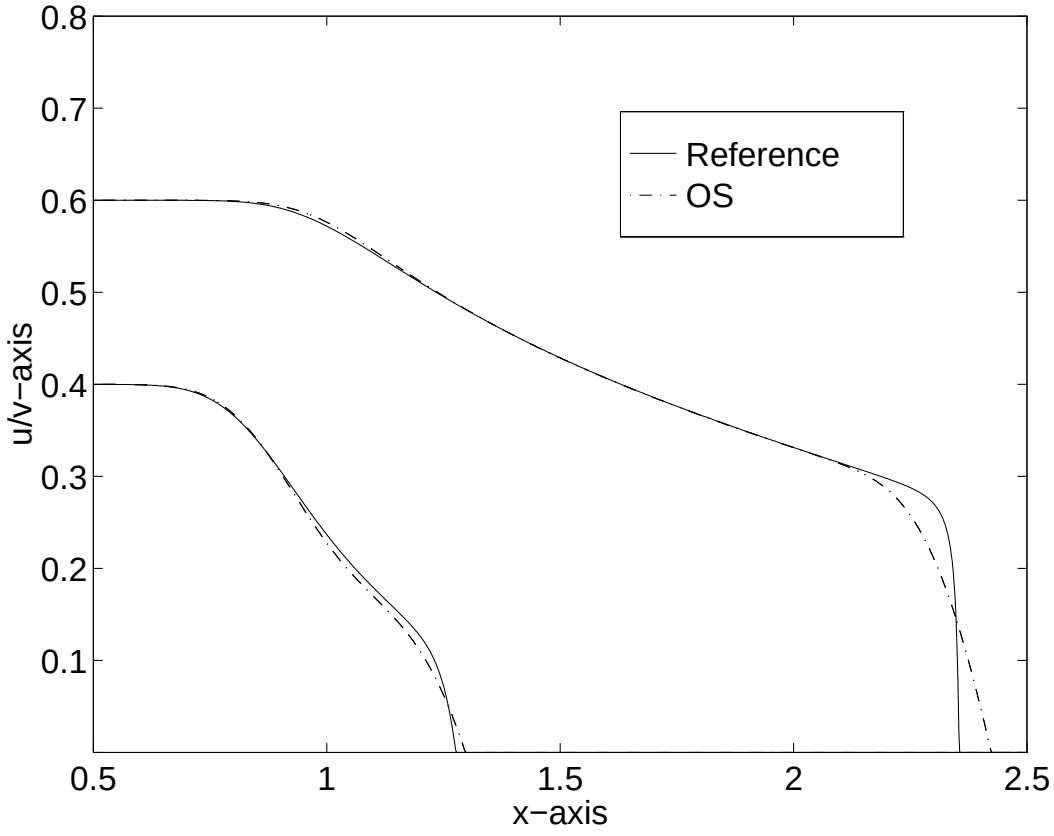
Ex. 2: Cauchy problem

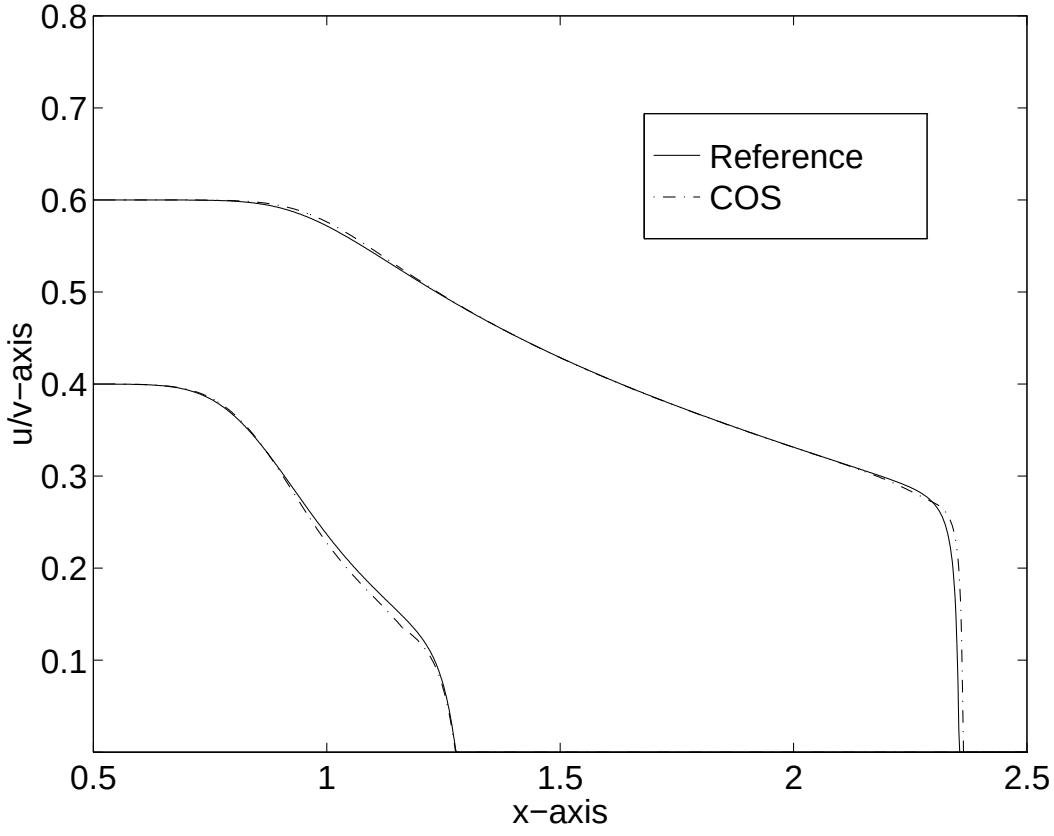
Initial data

$$u_0 = \begin{cases} 0.6 & \text{if} \quad x < 0.8, \\ 1.4 - x & \text{if} \quad 0.8 < x < 1.4, \\ 0.0 & \text{if} \quad x > 1.4, \end{cases} \quad \text{and}$$

$$v_0 = \begin{cases} 0.4 & \text{if} \quad x < 0.8, \\ 1.2 - x & \text{if} \quad 0.8 < x < 1.2, \\ 0.0 & \text{if} \quad x > 1.2. \end{cases}$$

$\epsilon = 0.01$, $t = 0.5$, $\Delta t = 0.5$ - one time step.


Reference
OS
u/v−axis
x−axis
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.5
1
1.5
2
2.5


Reference
COS
u/v−axis
x−axis
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.5
1
1.5
2
2.5

Ex. 3: Cauchy problem

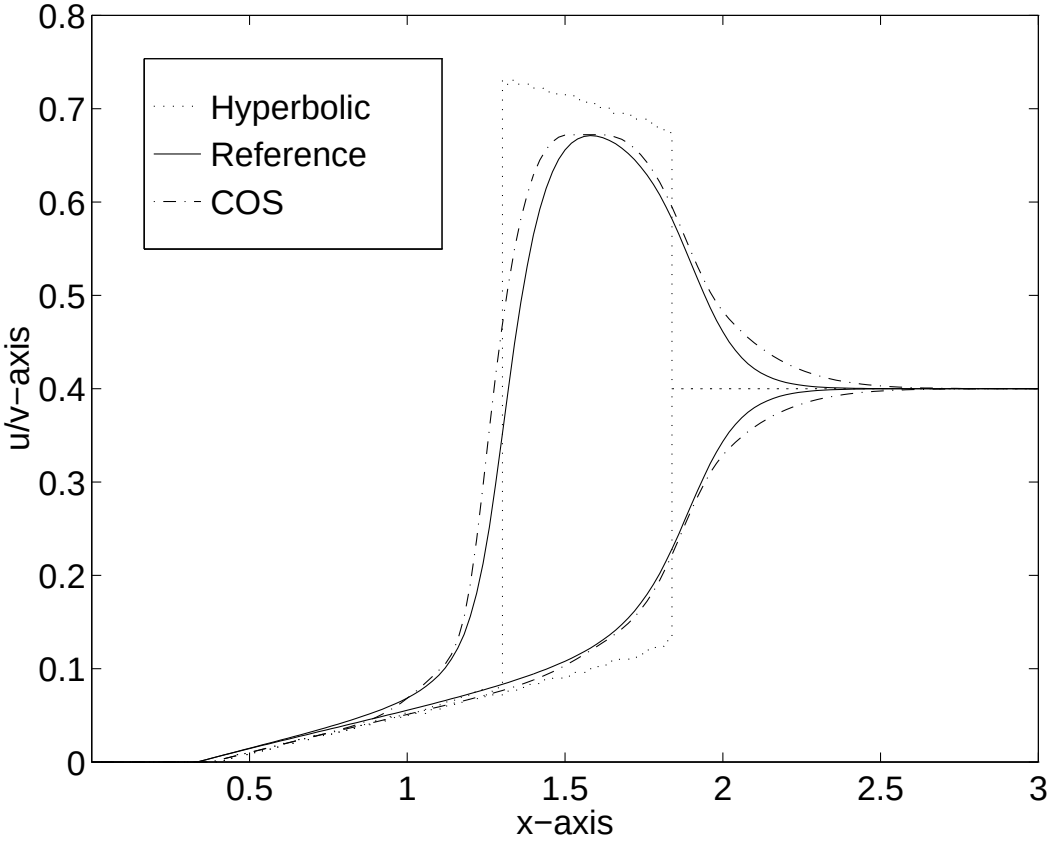
Initial data

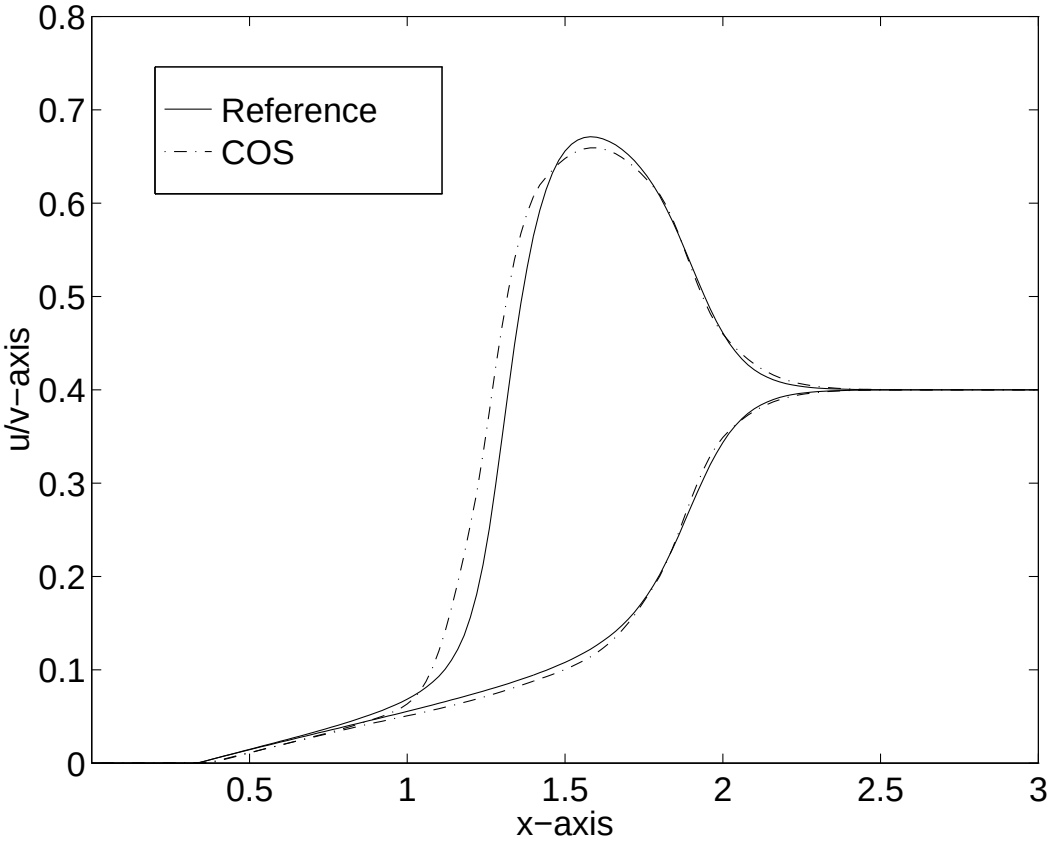
$$u_0 = v_0 = \begin{cases} 0.0 & \text{if} \quad x < 0.4, \\ x - 0.4 & \text{if} \quad 0.4 < x < 0.8, \\ 0.4 & \text{if} \quad x > 0.8. \end{cases}$$

$\epsilon = 0.1$, $t = 0.5$, one and three timesteps are used.

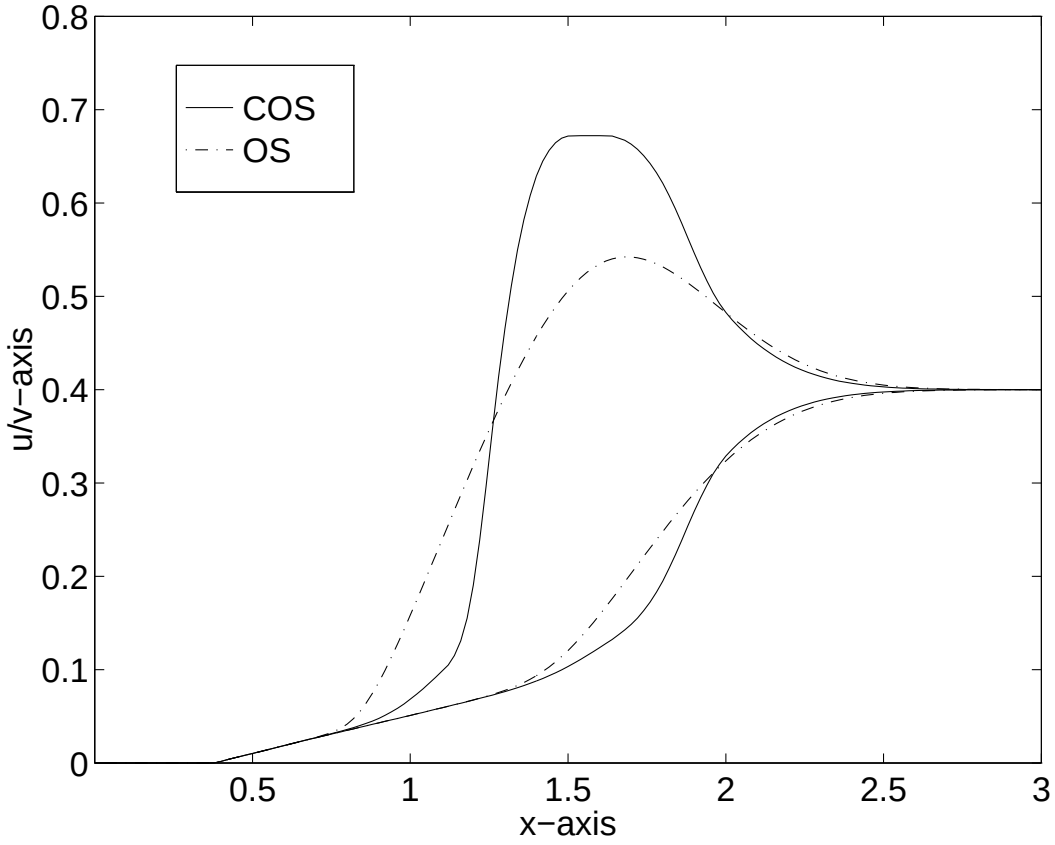
This data differs from Ex. 2 in the sense that the solution now is non-monotone.

One and three time steps





COS versus OS - one time step



SUMMARY

We have solved a 2×2 system of parabolic equations using the COS method:

- The hyperbolic step is solved with a front tracker.
- The front tracker solution generates the residual flux.
- The diffusion step is solved with the residual flux by a central finite difference scheme.

The numerical experiments showed that the COS method gives significantly better results than the OS method.